Conflict-free coloring with respect to a subset of intervals

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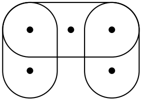
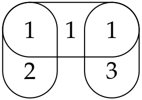
joint work with **Shakhar Smorodinsky**

Hypergraphs and colorings

Def. A *hypergraph* is a pair $(V, \mathcal{E})$, where $\mathcal{E}$ is a family of subsets of V. An element $e \in \mathcal{E}$ is called a *hyperedge*.



Def. A (vertex) coloring with k colors is a function $C\colon V \to \{1, \ldots, k\}$.



Conflict-free coloring and chromatic number

Def. A *conflict-free* (CF) coloring of $H = (V, \mathcal{E})$ is a coloring of H such that for every hyperedge $e \in \mathcal{E}$, there is a color in e which occurs exactly once in e.

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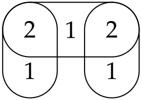

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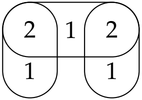
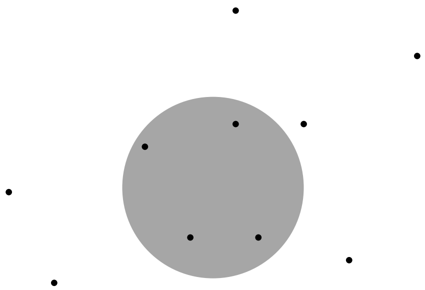


Def. The minimum k such that H has a CF-coloring with k colors is called the CF-*chromatic number* of H, denoted by $\chi_{\mathrm{cf}}(H)$.

Conflict-free coloring geometric hypergraphs

(Even, Lotker, Ron, Smorodinsky, 2003)
Given a set P of n points on the plane and a family $\mathcal{F}$ of geometric shapes, define a hypergraph $H = (V, \mathcal{E})$ as follows:

- the vertex set consists of the n points ($V = P$),
- a subset $S \subseteq P$ is in $\mathcal{E}$ iff $S = P \cap Q$, for some shape $Q \in \mathcal{F}$.



(application: frequency assignment in cellular networks)

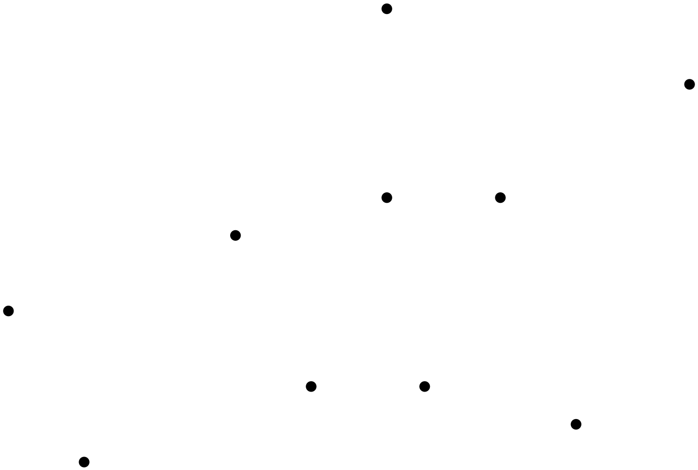
Motivation for conflict-free coloring

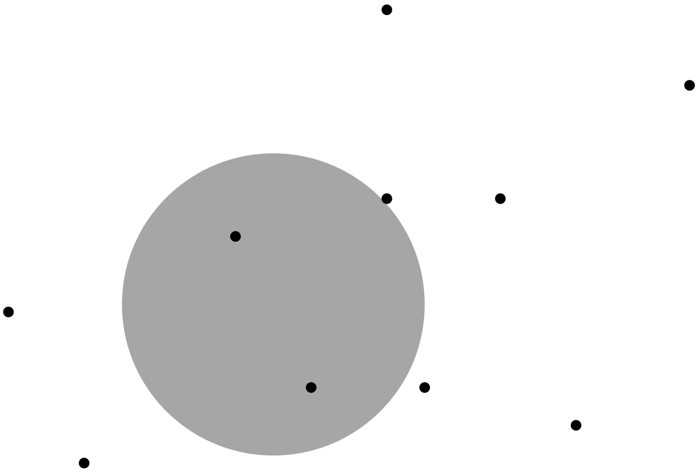
- Cellular networks consist of fixed position *base stations* (or antennas) that emit at a specific frequency, and *moving agents*.

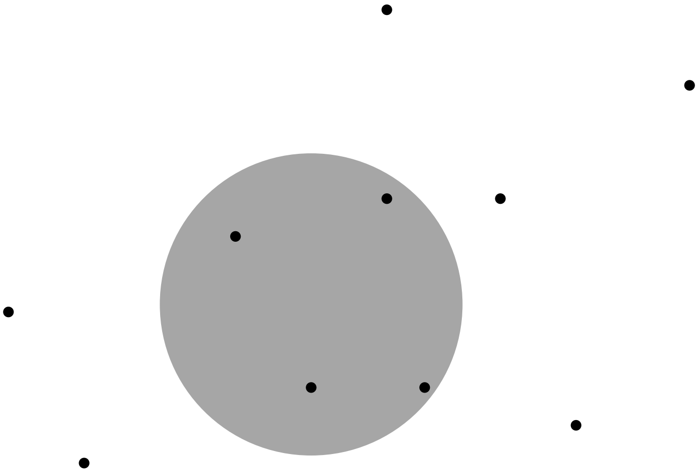
- Each moving agent has a range of communication that can be modeled by a shape (like a disk). The range includes a subset S of the base stations. We want each such S to contain a base station with unique frequency in S.

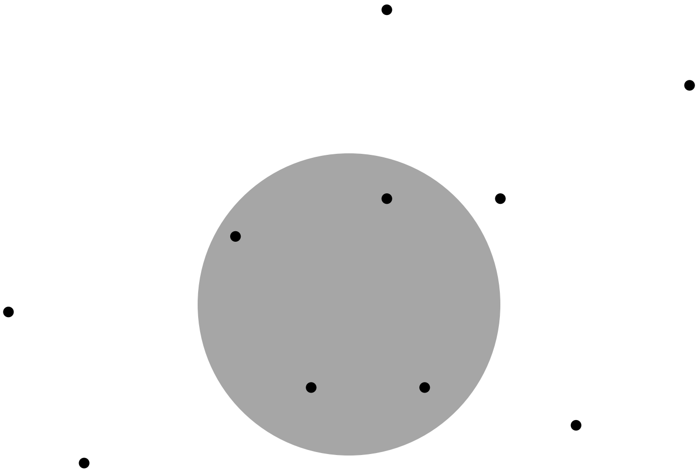
- Model: base stations $\rightarrow$ points, frequencies $\rightarrow$ colors

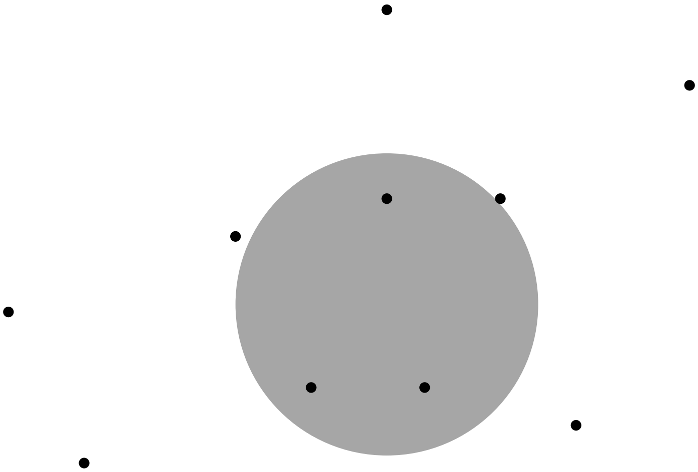
- The frequency spectrum is expensive. Therefore, we try to minimize frequency use, i.e., reuse frequencies as much as possible.

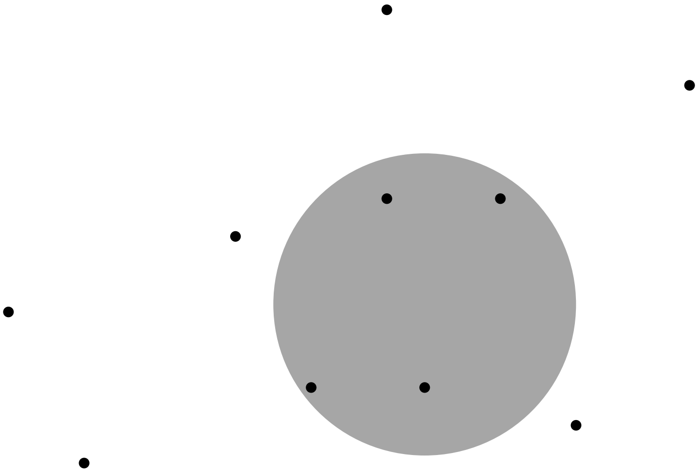


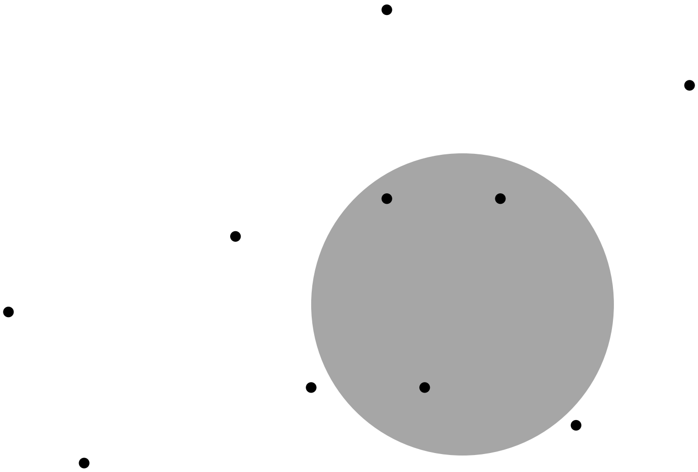


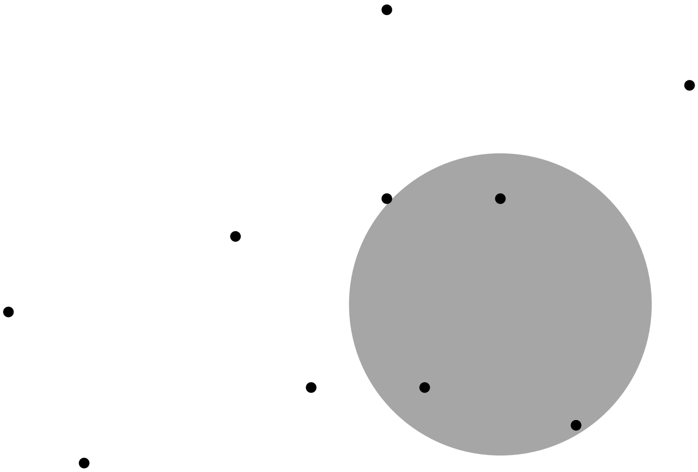


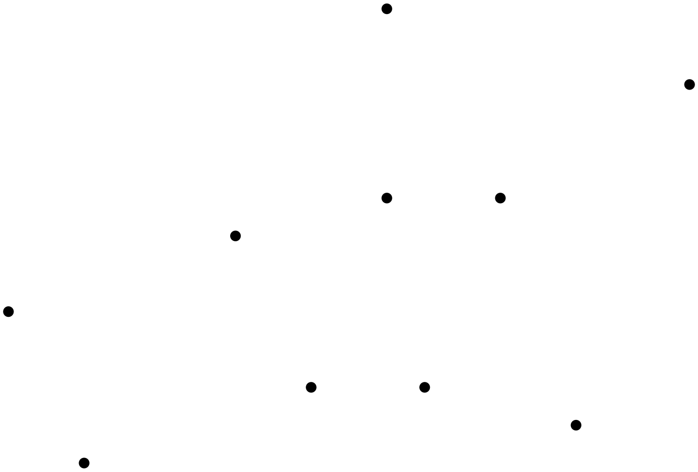


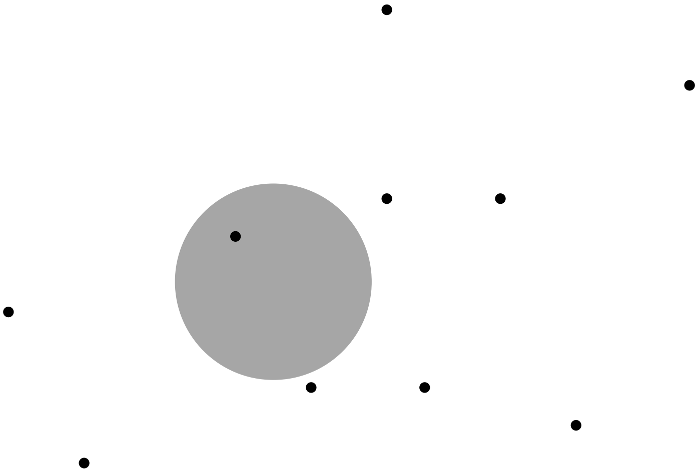


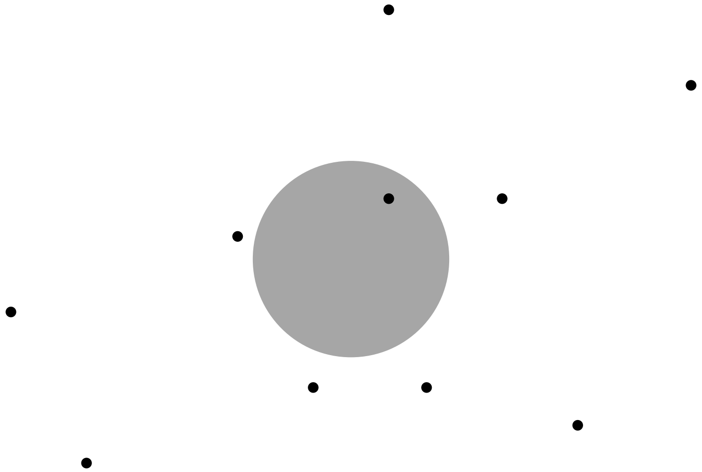


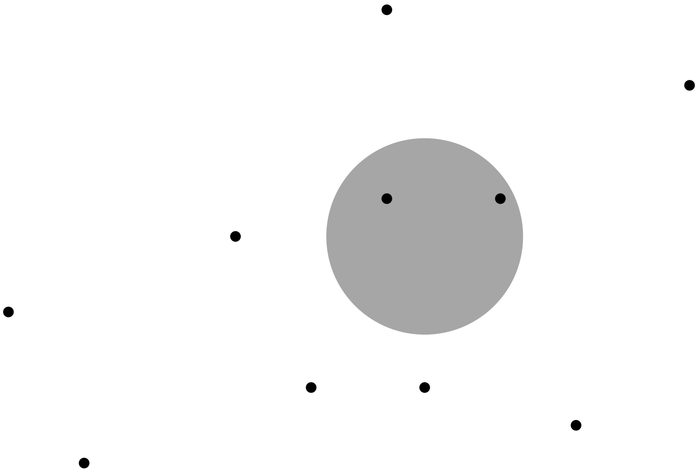






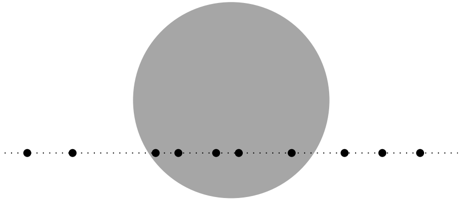






CF coloring collinear points w.r.t. disks

Special case: conflict-free coloring with respect to disks, when all n points to be colored are collinear.



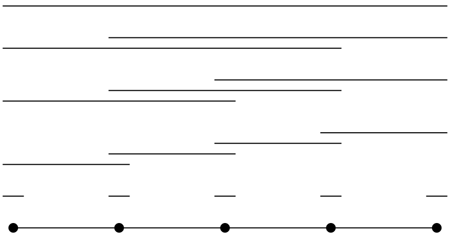
disks intersect with sets of consecutive points on the line.

equivalently, we have a hypergraph of n points on the real line with respect to all intervals.

We call this the *discrete interval hypergraph* H_n.
$H_n = (\{1, \ldots, n\}, \{[i, j] \cap \mathbb{N} \mid 1 \leq i \leq j \leq n\})$

CF coloring of the discrete interval hypergraph H_n

Example for H_5 ($n = 5$):



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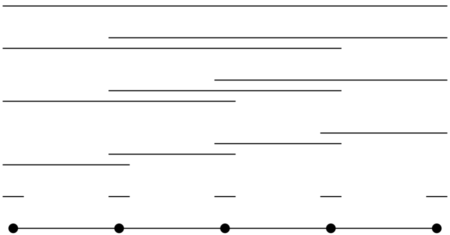

CF coloring: 1 2 3 1 2

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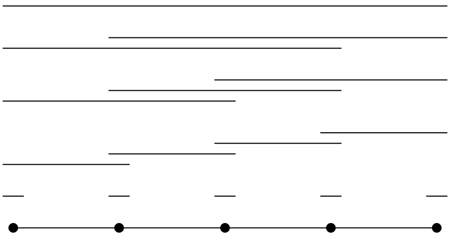

CF coloring:	1	2	3	1	2
illegal coloring:	1	2	1	2	3

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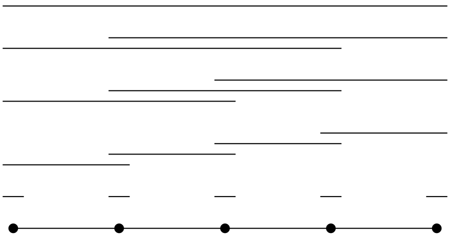
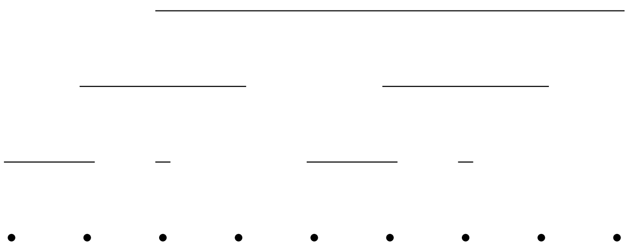


CF coloring: 1 2 3 1 2

illegal coloring: 1 2 1 2 3

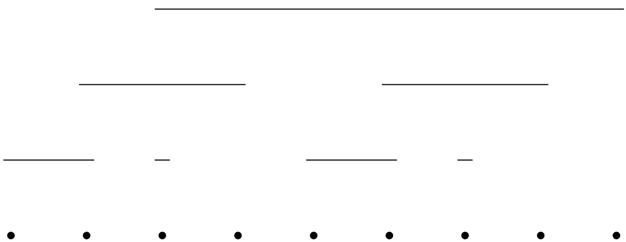
Conflict-free coloring w.r.t. a **subset** of intervals

Katz, Lev-Tov, Morgenstern in CCCG 2007 considered conflict-free coloring points with respect to a *subset* of all possible intervals.



Katz et al. claim a 4-approximation algorithm.

Our main result: a 2-approximation algorithm



Thm. There is a 2-approximation algorithm for conflict-free coloring with respect to a subset of intervals.

Moreover, we prove that there are tight instances, which are colored with twice the optimal number of colors by our algorithm.

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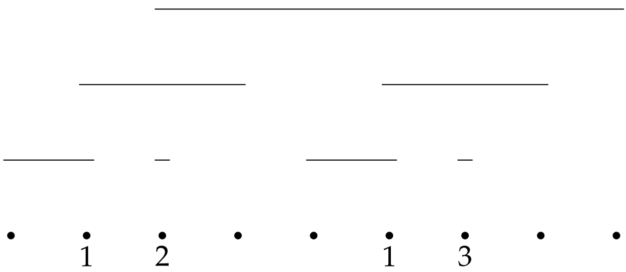

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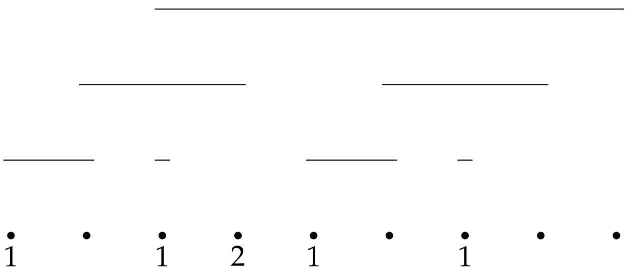

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A hitting-set algorithm for conflict-free coloring

$\ell \leftarrow 0$; $V^0 \leftarrow V$; $\mathcal{E}^0 \leftarrow \mathcal{E}$
while $\mathcal{E}^\ell \neq \emptyset$ **do**
  $S^\ell \leftarrow$ a minimal hitting set for $(V^\ell, \mathcal{E}^\ell)$
  color every $v \in V^\ell \setminus S^\ell$ with color ℓ
  $V^{\ell+1} \leftarrow S^\ell$
  $\mathcal{E}^{\ell+1} \leftarrow \{e \cap S^\ell \mid e \in \mathcal{E}^\ell \text{ and } |e \cap S^\ell| > 1\}$
  $\ell \leftarrow \ell + 1$
end while
if $V^\ell \neq \emptyset$ **then** color every $v \in V^\ell$ with color ℓ **end if**

Our other result: complexity

Def. decision problem CFSubsetIntervals:

> *"Given a subhypergraph $H = (\{1, \ldots, n\}, I)$ of the discrete interval hypergraph H_n and a natural number k, is it true that $\chi_{\mathrm{cf}}(H) \leq k$?"*

non-trivial only when $k < \lfloor \log_2 n \rfloor + 1$;

if $k \geq \lfloor \log_2 n \rfloor + 1$, answer is always yes, since $\chi_{\mathrm{cf}}(H_n) = \lfloor \log_2 n \rfloor + 1$.

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Thm. CFSubsetIntervals has a *quasipolynomial* time deterministic algorithm.

(quasipolynomial means $2^{O(\log^2 n)}$ in this case)

A non-deterministic $O(\log^2 n)$-space algorithm

Scan points from 1 to n.

For each point $t \in \{1, \dots, n\}$, try non-deterministically every color in $\{0, \dots, k\}$ and check if all intervals ending at t have the CF-property.

For every color $c \in \{0, \dots, k\}$, keep track of:
(a) the closest point to t colored with c in variable p_c
(b) the second closest point to t colored with c in variable s_c.

color c is occurring exactly one time in $[j, t] \in I$ if and only if $s_c < j \leq p_c$.

```
for c ← 0 to k do
   s_c ← 0
   p_c ← 0
end for
for t ← 1 to n do
   choose c non-deterministically from {0, ..., k}
   s_c ← p_c
   p_c ← t
   for j ∈ {j | [j, t] ∈ I} do
      IntervalConflict ← True
      for c ← 1 to k do
         if s_c < j ≤ p_c then
            IntervalConflict ← False
         end if
      end for
      if IntervalConflict then
         return NO
      end if
   end for
end for
return YES
```

Related work and open problems

PTAS? Polynomial time algorithm?

relationship of unique-maximum and conflict-free colorings w.r.t. a subset of intervals
(see also joint work with Géza Tóth, Balázs Keszegh and Dömötör Pálvölgyi, for UM/CF relationship in other hypergraphs)

strong conflict-free coloring (Luisa Gargano and Adele Rescigno)

common paper with the above authors in ISAAC 2012

Thank you!