Beyond polynomial approximation

Moderately exponential, subexponential and parameterized approximability

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Inapproximability

Inapproximability result

A statement claiming that a problem is not approximable within ratios better than some approximability level unless something very unlikely happens in complexity theory

- ***P*** $=$ ***NP***
- *Disapproval of the* ***ETH***
- ...

ETH

There exists an $\epsilon > 0$ such that no algorithm solves 3-SAT on n variables in time $2^{\epsilon n}$

Examples of inapproximability

- MAX INDEPENDENT SET or MAX CLIQUE inapproximable within ratios $\Omega\left(n^{-1}\right)$
- MIN VERTEX COVER within ratios smaller than 2
- MIN SET COVER within ratios $o(\log n)$
- MIN TSP within better than exponential ratios
- MIN COLORING within ratios $o(n)$
- ...

PCP (1)

A problem is in $\mathrm{PCP}_{\alpha,\beta}[q,p]$, if there exists a PCP verifier which uses q random bits, reads at most p bits in the proof and is such that:

- if the instance is positive, then there exists a proof such that V accepts with probability at least α
- if the instance is negative then, for any proof, V accepts with probability at most β

For every $\epsilon > 0$
3-SAT $\in \mathbf{PCP}_{1,\epsilon}[(1+o(1))\log n + O(\log(1/\epsilon)), O(\log(1/\epsilon))]$
(Dinur, (2007))

PCP (2): Linear **PCP** Conjecture

Linear **PCP** Conjecture (**LPC**) (Esoffier, Kim & P (2012))

Return

3-SAT $\in \mathbf{PCP}_{1,1/2}[\log|\phi| + c_1, c_2]$

$|\phi|$ the size of the 3-SAT instance (sum of lengths of clauses)

c_1 and c_2 constants

Parameterized complexity (1)

Parameterized decision problem

A pair (Π, k), where Π is a decision problem in the sense of (Garey & Johnson(1979)) and k a parameter (e.g., size of the solution, max degree, vertex cover number, ...)

The size of the solution is often called *standard* parameter

Fixed parameter tractable problem (FPT problem)

A problem (Π, k) is FPT if there exists an algorithm (called FPT algorithm) that decides it in time $O\left(f(k)|I|^c\right)$, for some computable function f and some constant c

The class of FPT problems will be denoted by **FPT**

Parameterized complexity (2)

The **W**-hierarchy

- Not all problems are in **FPT**
- Classes **W**$[\cdot]$ are hardness classes for parameterized complexity
- **W**$[1]$-hardness (proved by the so-called *parameterized reducibility*) is the parameterized equivalent of **NP**-completeness
- $\mathbf{FPT} \subseteq \mathbf{W}[1] \subseteq \mathbf{W}[2] \subseteq \ldots$ (proper containment is conjectured)

The key issue

Approximate optimal solutions of NP-hard problems within ratios "forbidden" to polynomial algorithms and with worst-case complexity provably better than the complexity of an exact computation

Do it

- For some forbidden ratio
- For any forbidden ratio

Generate a small number of candidates (1)

The key-idea

Generate a small number of candidate solutions (polynomially complete them, if necessary and possible) and return the best among them

Generate a small number of candidates (2): MAX INDEPENDENT SET

- Generate all the $\sqrt{n}$-subsets of V
- If one of them is independent, then return it
- Else return a vertex at random

Approximation ratio: **?**
Worst-case complexity: **?**
It works also for:

- MIN INDEPENDENT DOMINATING SET (Bourgeois, Escoffier & P (2010))
- CAPACITATED DOMINATING SET (Cygan, Pilipczuk & Wojtaszczyk (2010))

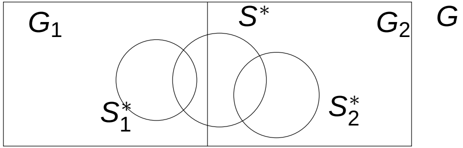
Divide & approximate (1)

The key-idea

Optimally solve a problem in a series of ("small") sub-instances of the initial instance

- Appropriately split the instance in a set of sub-instances (whose sizes are functions of the ratio that is to be achieved)
- Solve the problem in this set
- Compose a solution for the initial instance using the solutions of the sub-instances

Divide & approximate (2): Example for is and for $p/q = r = 1/2$



$$|S^*| \leqslant |S_1^*| + |S_2^*| \leqslant 2\max\{|S_1^*|, |S_2^*|\} \Longrightarrow \frac{\max\{|S_1^*|, |S_2^*|\}}{|S^*|} \geqslant \frac{1}{2}$$

Complexity: $O\left(\gamma^{n/2}\right)$, if $O\left(\gamma^{n}\right)$ the exact complexity of MAX INDEPENDENT SET

Divide & approximate (3): works also for ...

If $O\left(\gamma^{n}\right)$ the complexity for MAX INDEPENDENT SET

- MIN VERTEX COVER and MIN SAT: $(2-r)$-approximation in $O\left(\gamma^{rn}\right)$ ($O\left(\gamma^{rm}\right)$ for MIN SAT), for any r (Bourgeois, Escoffier & P (2011)), (Escoffier, P, & Tourniaire (2012a))
- MAX CLIQUE: r-approximation in $O\left(\gamma^{r\Delta}\right)$ (Δ the maximum degree of the input-graph), for any r (Bourgeois, Escoffier & P (2011))

Analogous results hold for MAX SAT (Escoffier, P, & Tourniaire (2012a))

Approximately pruning the search tree (1)

The key idea

Perform a branch-and-cut by allowing a “bounded error” in order to accelerate the algorithm (i.e., make the instance-size decreasing quicker than in exact computation by keeping the produced error “small”)

Approximately pruning the search tree (2): MAX INDEPENDENT SET

1. If $\Delta(G) \leqslant 7$, then approximate MAX INDEPENDENT SET polynomially;
2. else, branch on a vertex v with $d(v) \geqslant 8$ and either take it, remove its neighbors and two more vertices v_i, v_j such that $(v_i, v_j) \in E$, or do not take it

The above algorithm computes an $\frac{1}{2}$-approximation for MAX INDEPENDENT SET in time $O(1.185^n)$

Approximately pruning the search tree (3): MAX INDEPENDENT SET (cont.)

- If $\Delta(G) \leqslant 7$, approximation ratio $\frac{1}{2}$ (ratio $\frac{5}{\Delta(G)+3}$, (Berman & Fujito (1985)))
- If $\Delta(G) \geqslant 8$, we make an "error" of at most 1 vertex per vertex introduced in the solution (ratio $\frac{1}{2}$)

Complexity

$$T(n) \leqslant T(n-1) + T(n-11) + p(n) = O\left(1.185^n\right)$$

Approximately pruning the search tree (4): works also for . . .

- MIN SET COVER (Bourgeois, Escoffier & P (2009))
- BANDWIDTH (Cygan & Pilipczuk (2010))
- MIN and MAX SAT (Escoffier, P & Tourniaire (2012a))

(Escoffier, P & Tourniaire (2012b)) introduce some formalism for getting approximation schemata by approximation branching algorithms

Randomization

The key idea

Achieving ratio r with complexity better than $O\left(\gamma^{rn}\right)$

- Randomly split the graph into subgraphs in such a way that the problem at hand is to be solved in graphs G_i' of order $r'n$ with $r' < r$
- Compute the probability $\Pr[r]$ of an r-approximation
- Repeat splitting $N(r)$ times to get r-approximation with probability ~ 1 (in time $N(r)\gamma^{r'n}$)

It works for MAX INDEPENDENT SET, MIN VERTEX COVER, MIN SET COVER, . . . (Bourgeois, Escoffier & P (2009,2011))

What is it?

A parameterized (with parameter k) problem Π is r-approximable, if there exists an algorithm taking as inputs an instance I of Π and k and either computes a solution smaller or greater than rk, or returns "no there is no solution of value at most or at least k"

The techniques described previously for moderately exponential approximation remain valid

Some basic questions

1. Given an **FPT** problem can we get good parameterized approximation algorithms? Can we do this for *any* **FPT** problem?
2. Can a $\mathbf{W}[\cdot]$ problem be well-approximable in parameterized time?
3. Does a highly inapproximable (in polynomial time) problem admit a good parameterized approximation algorithm?

Question 1 (1)

MIN VERTEX COVER

Parameterized approximation schemata in time $O\left(\gamma^{(1-f(\epsilon))k}\right)$ (if $O\left(\gamma^{k}\right)$ is its – exact – parameterized time) by:

- divide & approximate (Bourgeois, Escoffier & P (2008))
- approximate pruning of the search tree (Brankovic & Fernau (2010))
- a kind of approximability preserving reduction, called fidelity preserving transformation (Fellows, Kulik, Rosamond & Shachnai (2012))

Question 1 (2)

EDGE DOMINATING SET

Similar results by approximate pruning of the search tree (Escoffier, Monnot, P & Xiao (2012))

BUT

It seems not true for any **FPT** problem

Question 2 (1)

MAX (resp. MIN) k-CUT
Given G and k, find the partition of V into (V_1, V_2) with $|V_1| = k$ that maximizes (resp., minimizes) the cut (V_1, V_2)

Two "natural" parameters: k and the size ℓ of the cut (standard parameter):

- **W[1]**-hard with respect to k (Cai (2007))
- **FPT** with respect to ℓ (Bonnet, Escoffier, P & Tourniaire (2012))

Question 2 (2)

MAX k-CUT admits a parameterized approximation schema with respect to k (Bonnet, Escoffier, P & Tourniaire (2012))

- There exists an $O\left(\Delta^k\right)$ exact algorithm for MAX k-CUT (Bonnet, Escoffier, P & Tourniaire (2012))
- Consider as a candidate solution the k larger-degree vertices and take $\epsilon = k^2/\Delta$

Question 2 (3)

The same remains true for MIN k-CUT that is:

- **W[1]**-hard with respect to k (Cai (2007))
- of unknown status with respect to ℓ (Bonnet, Escoffier, P & Tourniaire (2012))

BUT

We don't know problems having this behaviour when considering standard parameter

Question 3 (1)

MIN INDEPENDENT DOMINATING SET is not $g(k)$-approximable in FPT time, unless $\mathbf{FPT} = \mathbf{W}[2]$ (Downey, Fellows & McCartin (2006))

What about other paradigmatic problems like MAX INDEPENDENT SET, MIN DOMINATING SET, ...?

Question 3 (2)

LPC ▸ LPC implies the following (Escoffier, Kim & P (2012))

Hypothesis H

Under **ETH**, there exists $r < 1$ such that, for every $\epsilon > 0$, it is impossible to distinguish between instances of MAX 3-SAT with m clauses where at least $(1 - \epsilon)m$ are satisfiable from instances where at most $(r + \epsilon)m$ clauses are satisfiable, in time $O\left(2^{o(m)}\right)$

Question 3 (3)

MAX INDEPENDENT SET

Under **H** and **ETH**, for any $\rho \in (0, 1)$, no approximation parameterized algorithm for MAX INDEPENDENT SET (i.e., an algorithm that runs in time $f(k)p(n)$ for some function f and some polynomial p) can guarantee approximation ratio ρ (Escoffier, Kim & P (2012))

Actually, a somewhat stronger result holds

Under **H** and **ETH**, for every $\epsilon > 0$, no parameterized approximation algorithm for MAX INDEPENDENT SET running in time $f(k)n^{o(k)}$ can achieve approximation ratio $r + \epsilon$

Question 3 (4)
MIN DOMINATING SET

Under **H** and **ETH**, for every $\epsilon > 0$, no approximation algorithm running in time $f(k)n^{o(k)}$ can achieve approximation ratio smaller than $2 - r - \epsilon$ for MIN DOMINATING SET (Escoffier, Kim & P (2012))

The same holds for MIN SET COVER modulo the substitution of n by m (the size of the set-system)

The key-issue

Approximate optimal solutions of NP-hard problems within ratios "forbidden" to polynomial algorithms and with *subexponential* worst-case complexity

An anecdotal observation

A very large number of **NP**-hard problems (at least those whose solutions are subsets of the input data) are solvable either in subexponential, or in FPT time!!!!

Proof?

MAX INDEPENDENT SET

Under **ETH**, for any $r > 0$ and any $\delta > 0$, no r-approximation algorithm for MAX INDEPENDENT SET van run in time $O(2^{n^{1-\delta}})$ (Escoffier, Kim & P (2012))

Remark

MAX INDEPENDENT SET is approximable within any ratio $1/f(n)$ for any unbounded function $f(n)$ in time $O\left(\binom{n}{n/f(n)}\right)$

Just take the largest independent set between the subsets of V with cardinality at most $n/f(n)$

MIN COLORING and MIN VERTEX COVER

MIN COLORING

Under **ETH**, for any $r > 1$ and any $\delta > 0$, no r-approximation algorithm for MIN COLORING can run in time $O(2^{n^{1-\delta}})$ (Escoffier, Kim & P (2012))

MIN VERTEX COVER

Under **ETH**, for any $r > 0$ and any $\delta > 0$, no $(7/6 - \epsilon)$-approximation algorithm for MIN VERTEX COVER can run in time $O(2^{n^{1-\delta}})$ (Escoffier, Kim & P (2012))

The same holds nor for MIN SAT modulo substitution of n by m (the number of clauses in the formula)

And under **H**?

If **H** holds, then all the previous results become valid for any (non-trivial) time $2^{o(n)}$
(Escoffier, Kim & P (2012))

Structure of moderately exponential and subexponential approximation (1)

- Moderately exponential and subexponential approximation preserving reductions?
- Is it possible to get moderately exponential inapproximability results? Of what kind? Under what complexity conditions?
- Are there natural problems having some kind of self-improvement property saying, for instance, that r-approximability in time $O(\gamma^n)$ implies exact solution in time $O(\gamma^n)$?

Structure of moderately exponential and subexponential approximation (2)

- Is **ETH** sufficient for showing inapproximability, within any ratio, of MAX INDEPENDENT SET, MIN DOMINATING SET, MIN COLORING in $O\left(2^{o(n)}\right)$? Within certain ratios?
- Can we prove subexponential inapproximability of MIN VERTEX COVER or MAX 3-SAT within certain ratios?
- Is it possible to show subexponential inapproximability results under some hypothesis like "there exists some r such that MAX 3-SAT is inapproximable within ratio r in time $O\left(2^{o(n)}\right)$ (an "**AETH**")?

Structure of parameterized approximation

- Parameterized approximability preserving reductions (how an “approximation kernel” looks like?)
- is **LPC** (and, consequently, **H**) true?
- Do we need a “parameterized” PCP? How would it look like?

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