Computing the volume of the discriminant polytope

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Discriminants in high school

Example

$$\left.\begin{array}{l} f(x) = ax^2 + bx + c = 0 \\ f'(x) = 2ax + b = 0 \end{array}\right\} \quad \Delta = b^2 - 4ac$$

- Δ vanishes iff f has a multiple root

Another example

A degree 5 polynomial on one variable

$$
\begin{aligned}
f(x) &= ax^5 + bx^4 + cx^3 + dx^2 + ex + g = 0 \\
f'(x) &= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e = 0
\end{aligned}
$$

- Elimination theory reduces the computation of Δ to the computation of a 9×9 determinant.

$$
\begin{array}{l}
\Delta = -2050a^2g^2bedc + 356abed^2c^2g - 80b^3ed^2cg + 18dc^3b^2g \\
e - 746agdcb^2e^2 + 144ab^2e^4c - 6ab^2e^3d^2 - 192a^2be^4d - 4d^2ac \\
{}^3e^2 + 144d^2a^2ce^3 - 4d^3b^3e^2 - 4c^3e^3b^2 - 80abe^3dc^2 + 18b^3e^3 \\
dc + 18d^3acbe^2 + d^2c^2b^2e^2 - 27b^4e^4 - 128a^2e^4c^2 + 16ac^4e^3 - 27 \\
a^2d^4e^2 + 256a^3e^5 + 3125a^4g^4 + 160a^2gbe^3c + 560a^2gdc^2e^2 + 1020 \\
a^2gbd^2e^2 + 160ag^2b^3ed + 560ag^2d^2cb^2 + 1020ag^2b^2c^2e - 192 \\
b^4ecg^2 + 24ab^2ed^3g + 24abe^2c^3g + 144b^4e^2dg - 6b^3e^2c^2g + 14 \\
4dc^2b^3g^2 - 630dac^3bg^2 - 630d^3a^2ceg - 72d^4acbg - 72dac^4e \\
g - 4d^3c^2b^2g - 1600ag^3cb^3 - 2500a^3g^3be - 50a^2g^2b^2e^2 - 3750a^3 \\
g^3dc + 2000a^2g^3db^2 + 2000a^3g^2ce^2 + 825a^2g^2d^2c^2 + 2250a^2g^3b \\
c^2 + 2250a^3g^2ed^2 - 900a^2g^2bd^3 - 900a^2g^2c^3e - 36agb^3e^3 - 1600 \\
a^3ge^3d + 16d^3ac^3g - 128d^2b^4g^2 + 16d^4b^3g - 27c^4b^2g^2 + 108ac^5 \\
g^2 + 108a^2d^5g + 256b^5g^3
\end{array}
$$

- The number of Δ terms increases exponentially with the degree!

One more example

- A system of two polynomials on two variables

$$\begin{aligned} f_1 &= ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + g \\ f_2 &= hx_1^2 + ix_1x_2 + jx_2^2 + kx_1 + lx_2 + m \end{aligned}$$

- The condition of the two quadrics f_1, f_2 to be tangent is expressed by the Δ of

$$\begin{aligned} f = \quad & ax_1^2x_3 + bx_1x_2x_3 + cx_2^2x_3 + dx_1x_3 + ex_2x_3 + gx_3 + hx_1^2x_4 + ix_1x_2x_4 \\ & + jx_2^2x_4 + kx_1x_4 + lx_2x_4 + mx_4 \end{aligned}$$

- Δ is of degree 12 and has 3210 monomials!

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\end{aligned}
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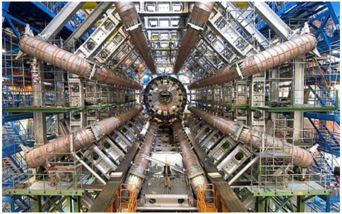
Instrumental in dark matter searches at the CERN Large Hadron Collider



What is the discriminant?

Definition

Given $f(x) = \sum_{a \in A} c_a x^a$, where $A \subset \mathbb{Z}^d$, $x = (x_1, \ldots, x_d) \in (\mathbb{C}^*)^d$.

The discriminant is the unique (up to sign) irreducible polynomial Δ with integer coefficients in the unknowns c_a which vanishes iff f has a multiple root, i.e.,

$$\Delta = 0 \quad \Leftrightarrow \quad \exists x^* \in (\mathbb{C}^*)^d \quad \text{s.t.} \quad f(x^*) = \frac{\partial f}{\partial x_1}(x^*) = \cdots = \frac{\partial f}{\partial x_d}(x^*) = 0$$

What is the discriminant polytope?

Definition

- The Newton polytope of f, $N(f)$, is the convex hull of the set of exponents of its monomials with non-zero coefficient.
- The discriminant polytope is $N(\Delta)$.

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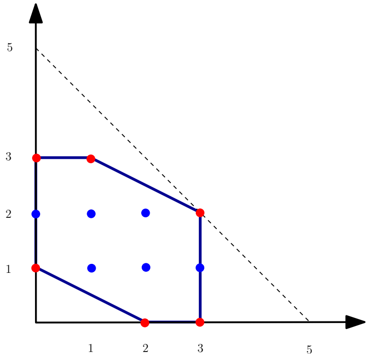
Example

$$f(x_1, x_2) = 8x_2 + x_1x_2 - 24x_2^2 - 16x_1^2 + 220x_1^2x_2 - 34x_1x_2^2 - 84x_1^3x_2 + 6x_1^2x_2^2 - 8x_1x_2^3 + 8x_1^3x_2^2 + 8x_1^3 + 18x_2^3$$



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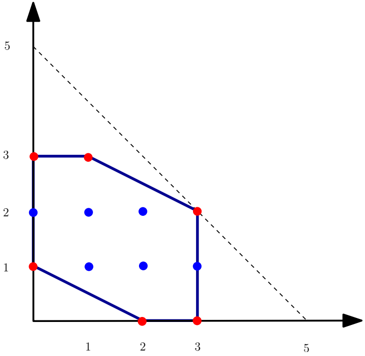
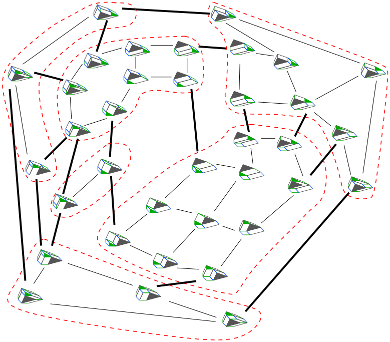
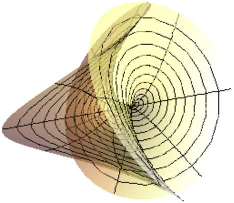


- We consider d fixed and $d > 4$. If $n := |A|$, $dim(N(\Delta)) = n - d + 1$.
- Knowing $N(\Delta)$, reduces the computation of Δ to a linear algebra problem!

Discriminant polytope: Motivation

- Geometry: equival. classes of the polytope of all triangulations
- Algebra: generalizes the notion of degree of the discriminant
- Applications: discriminant computation, CAD: implicitization of parametric hypersurfaces



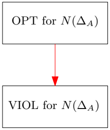


Enneper's Minimal Surface

Existing work

- Theory of resultant and discriminant polytopes [GelfandKapranovZelevinsky'94]
- TOPCOM [Rambau '02] computes all vertices of the polytope of all triangulations.
- [Emiris, F, Konaxis, Peñaranda SoCG'12] algorithm for computing resultant polytopes (`respol`) (extended by [Emiris, F, Dickenstein] to discriminant polytopes).
- Complexity of computing $N(\Delta)$ dominated by convex hull ([Chazelle'91]) $O(|V|^{\lfloor n/2 \rfloor})$, $V =\{$vertices of $N(\Delta)\}$.
- Tropical geometry [Sturmfels-Yu '08]: algorithms for resultant polytope (`GFan`) [Jensen-Yu '11] and discriminant polytope vertices (`TropLi`) [Rincón'12].

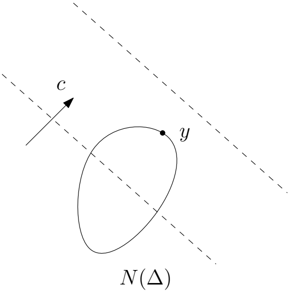
Polytope oracles



OPT oracle [EFKP'12]

Given $c \in \mathbb{R}^n$, find $y \in N(\Delta) \in \mathbb{R}^n$: maximize $c^T y$, or assert that $N(\Delta)$ is empty.

Complexity: dominated by convex hull $O(n^{\lfloor d/2 \rfloor})$

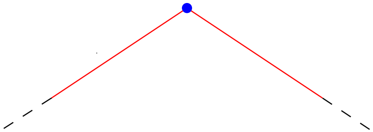


VIOL oracle

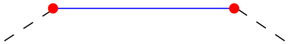
Given $c \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, does $c^T x \leq \gamma$ holds $\forall x \in N(\Delta)$? If not, find $y \in N(\Delta)$ with $c^T y > \gamma$.

Polytope duality

- $N(\Delta)^* := \{(z^T, \lambda)^T \in \mathbb{R}^{n+1} \ : \ z^T x \leq \lambda \text{ for all } x \in N(\Delta)\}$



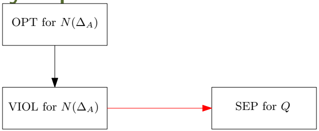
$N(\Delta)$

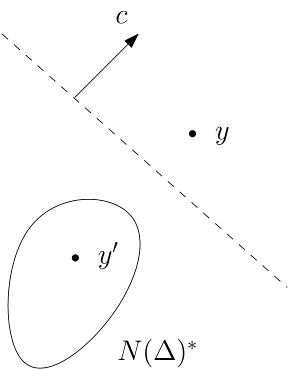


$N(\Delta)^*$

- $N(\Delta)^{**} = N(\Delta)$

Polytope oracles and duality





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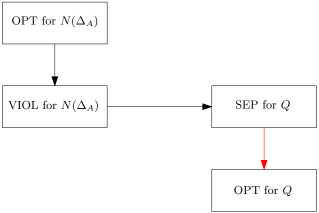
SEP oracle

Given $y \in \mathbb{R}^n$, does $y \in N(\Delta)^*$? If not, find $c \in \mathbb{R}^n$ s.t. $c^T y > max\{c^T x \mid x \in P\}$.

Optimization from Separation



polynomial time optimization algorithms

- Ellipsoid [GrötschelLovászSchrijver'88]
- Vaidya's [Vaidya'89].
- Centralized Splitting [Levin'65]
- random walk [BertsimasVempala'04]

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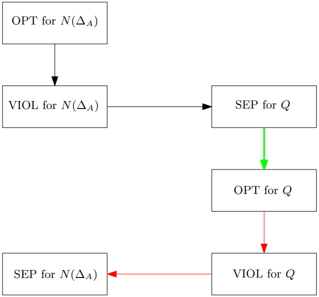

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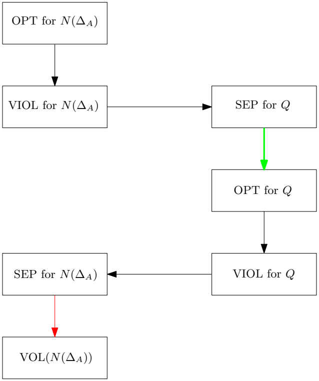
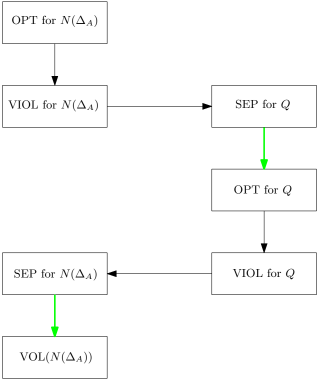
The volume problem



- Convex polytope $Q \subseteq \mathbb{R}^n$, computing volume is $\#P - hard$
- Given separation oracle for Q, $\exists$ randomized algorithm s.t. approximate Q volume (arbitrary accuracy) in $O^*(n^4)$ [LovászVempala'06].
- The implementation by [Lovász et al.'04] run only with hypercubes in dimension < 10.

The volume of $N(\Delta)$



Theorem

Given a polynomial with d variables and n monomials we can compute an approximation of the volume of $N(\Delta)$ in $O^*(n^{\lfloor 2(d+3) \rfloor} L)$.

(where $L = log \frac{R}{r}$ and $N(\Delta)$ is contained in an axis-aligned cube of width R and contains a cube of width r)

Works for every polytope defined by an optimization oracle s.a. secondary, resultant polytopes . . .

Ongoing and future work

- Implement the volume computation algorithm for $N(\Delta)$
- Approximation of $N(\Delta)$

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Thank You !