Computation of Nash Equilibria in Bimatrix Games

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Monday, August 27 2012

Skeleton of the Talk

1. Bimatrx Games Preliminaries
   - Complexity of 2NASH
   - Formulations for 2NASH
   - The algorithm of Lemke & Howson
2. Polynomial-time Tractable Subclasses
3. Approximability of 2NASH
   - Theoretical Analysis
   - Experimental Study
4. Conclusions

Bimatrix Games: Representation

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- m (n) alternative **actions** for ROW (col) player.

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- Representation: By the $m \times n$ **bimatrix** of normalized payoffs:

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	1	ooo	j	ooo	n
1	*,*	*,*	*,*	*,*	*,*
ooo	*,*	*,*	*,*	*,*	*,*
k	*,*	*,*	$R_{k,j}, C_{k,j}$	*,*	*,*
ooo	*,*	*,*	*,*	*,*	*,*
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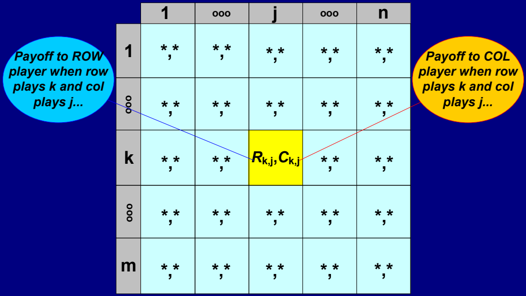

- Size of representation: $O(m \times n)$ rational numbers.

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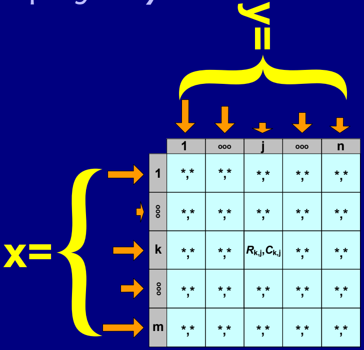

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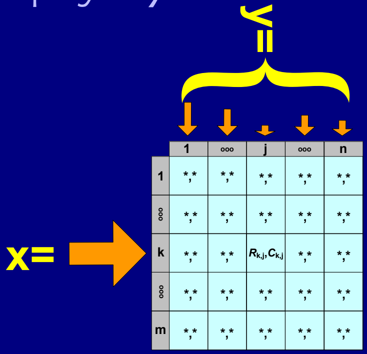

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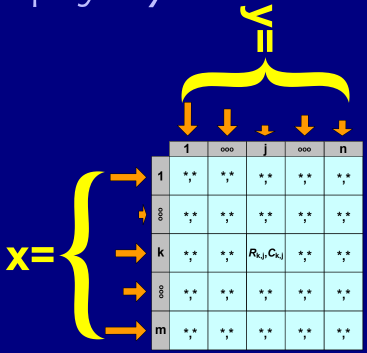

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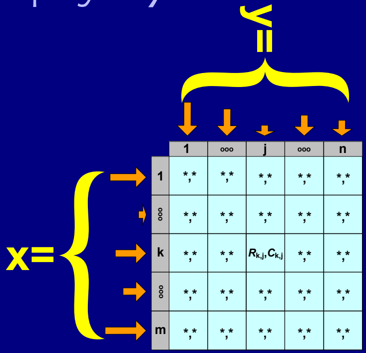

- Each player is interested in optimizing the expected value of her own payoff, as a function of the announced **strategies profile** for both players $(\mathbf{x}, \mathbf{y})$:
  - ROW player: $\mathbf{x}^T R \mathbf{y} = \sum_{i \in [m]} \sum_{j \in [n]} x_i \cdot R_{i,j} \cdot y_j$.
  - column player: $\mathbf{x}^T C \mathbf{y} = \sum_{i \in [m]} \sum_{j \in [n]} x_i \cdot C_{i,j} \cdot y_j$.

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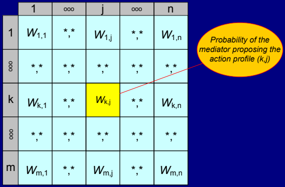

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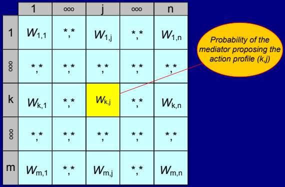

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ooo	*,*	*,*	*,*	*,*	*,*
m	$W_{m,1}$	*,*	$W_{m,j}$	*,*	$W_{m,n}$

Probability of the mediator proposing the action profile (k,j)

- The mediator *recommends* via private channels the corresponding action per player in the chosen profile, without revealing it to the opponent. Each player *freely chooses* an action, *already knowing* mediator's recommendation to her.

- The expected value of the payoff to each player, for a correlated strategy $\mathbf{W}$ of the mediator, is:

ROW player: $\sum_{k \in [m]} \sum_{j \in [n]} W_{k,j} R_{k,j}$ column player: $\sum_{k \in [m]} \sum_{j \in [n]} W_{k,j} C_{k,j}$

Uncorrelated vs Correlated Strategies

- Any *uncorrelated strategies-profile* $(\mathbf{x}, \mathbf{y}) \in \Delta([m]) \times \Delta([n])$ for the players, induces an **equivalent** *correlated strategy* (wrt to expected payoffs) for the mediator $\mathbf{W} = \mathbf{x} \cdot \mathbf{y}^T \in \Delta([m] \times [n])$.

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- Some correlated strategies **cannot be decomposed** into equivalent uncorrelated strategies profiles for the players. Eg, in the following (chicken) game, the correlated strategy $[(C, c) : 1/3, \ (C, d) : 1/3, \ (D, c) : 1/3, \ (D, d) : 0]$ is not decomposable into a pair of independent strategies for the players:

ROW \ col	dare	chicken
DARE	0 , 0	7 , 2
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Popular Solution Concepts (I)

DEFINITION: Approximations of Nash Equilibrium

For a *normalized* bimatrix game $\langle R, C \rangle$, an *uncorrelated strategies profile* $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is:

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- **ε-approximate Nash equilibrium** (ε-NE) iff no player can improve her expected payoff more than an additive term of $\varepsilon \geq 0$, by changing *unilaterally* her strategy, against the given strategy of the opponent: $\forall \mathbf{x} \in \Delta([m]),\ \forall \mathbf{y} \in \Delta([n]),$
$$\bar{\mathbf{x}}^T R \bar{\mathbf{y}} \geq \mathbf{x}^T R \bar{\mathbf{y}} - \varepsilon \ \wedge \ \bar{\mathbf{x}}^T C \bar{\mathbf{y}} \geq \bar{\mathbf{x}}^T C \mathbf{y} - \varepsilon$$

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- **ε-well-supported approximate Nash equilibrium** (ε-WSNE) iff no player assigns *positive probability mass* to any of her actions whose expected payoff against the given strategy of the opponent is less than an additive term $\varepsilon \geq 0$ from the maximum expected payoff: $\forall i \in [m],\ \forall j \in [n]$,

$$[\bar{x}_i > 0 \ \rightarrow\ \mathbf{e_i}^T R \bar{\mathbf{y}} \geq \max(R\bar{\mathbf{y}}) - \varepsilon] \ \wedge\ [\bar{y}_j > 0 \rightarrow \bar{\mathbf{x}}^T C \mathbf{e_j} \geq \max(\bar{\mathbf{x}}^T C) - \varepsilon]$$

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3. For $\varepsilon = 0$, ε-NE $\leftrightarrow$ ε-WSNE: These are **(exact) Nash equilibria**, where each player assigns all her probability mass to actions that are payoff maximizers (**best responses**) against the given strategy of the opponent.

4. Approximate Nash equilibria are invariant under shifts but are affected by scalings of the payoff matrices. Exact Nash equilibria are also invariant under positive scalings.

Popular Solution Concepts (II)

DEFINITION: Correlated Equilibrium

A *correlated strategy* $\bar{\mathbf{W}} \in \Delta([m] \times [n])$ (for the mediator) is a **correlated equilibrium** iff no player can improve her expected payoff by *unilaterally* ignoring the recommendation of the mediator, given that the opponent will adopt her own recommendation by the mediator: $\forall i, k \in [m],\ \forall j, \ell \in [n]$,

$$\sum_{j \in [n]} (R_{i,j} - R_{k,j}) \bar{W}_{i,j} \geq 0 \text{ and } \sum_{i \in [m]} (C_{i,j} - C_{i,\ell}) \bar{W}_{i,j} \geq 0$$

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Remarks:

1. The space of correlated equilibria is a polytope!!!
2. The existence of a mediator, although helpful in having the players' actions coordinated, raises serious concerns wrt implementation, such as trust, manipulability, objectivity, etc.

Popular Solution Concepts (III)

DEFINITION: MAXMIN Strategies

A profile of (uncorrelated) strategies $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is **MAXMIN profile** iff:

$$\bar{\mathbf{x}} \in \arg\max_{\mathbf{x}\in\Delta([m])} \min_{\mathbf{y}\in\Delta([n])} \mathbf{x}^T R \mathbf{y} \quad = \quad \arg\max_{\mathbf{x}\in\Delta([m])} \min_{j\in[n]} \mathbf{x}^T R \mathbf{e_j}$$

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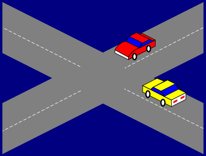
Remarks:

1. Efficiently computable via Linear Programming.
2. Extremely *pessimistic predictions*: It is possible that a MAXMIN profile is "too far" from any notion of approximate equilibrium (not only as points, but also wrt the approximation guarantee).

AN EXAMPLE: Chicken Game



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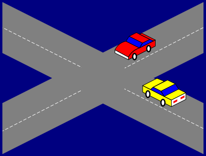

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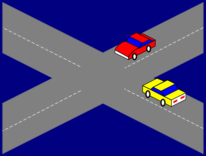

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- An additional *mixed* Nash equilibrium:
  $\left(\left[D : \frac{1}{3},\ C : \frac{2}{3}\right],\ \left[d : \frac{1}{3},\ c : \frac{2}{3}\right]\right)$.
  Expected payoff per player: $0 \cdot \frac{1}{9} + 2 \cdot \frac{2}{9} + 7 \cdot \frac{2}{9} + 6 \cdot \frac{4}{9} = \frac{14}{3}$.

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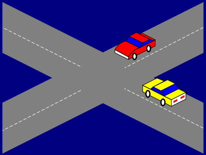

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- One more (extreme) (which are the others?) *correlated equilibrium*, external to the set $conv(NE(R, C))$:
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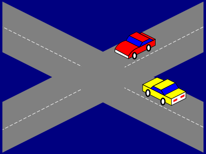

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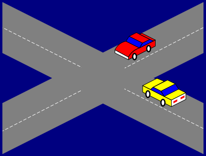

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- *MAXMIN* profile? (C, c) with payoff 6 per player.

Complexity of Equilibria

Hardness of Computing Equilibria

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- Correlated equilbria: Their space is a polytope described by a polynomial number of constraints.

$$CE(R,C) = \left\{ \begin{array}{l} \bar{\mathbf{W}} \in \Delta([m]\times[n]) : \\ \quad \forall i,k \in [m],\ \sum_{j\in[n]}(R_{i,j} - R_{k,j})\bar{W}_{i,j} \geq 0 \\ \quad \forall j,\ell \in [n],\ \sum_{i\in[m]}(C_{i,j} - C_{i,\ell})\bar{W}_{i,j} \geq 0 \end{array} \right\}$$

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- MAXMIN profiles: Again polynomial-time solvable, via Linear Programming.

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- Nash equilibria: Harder case, $\mathcal{PPAD}-$hard problem, even for bimatrix games, or arbitrarily good approximations!!!

Solving Bimatrix Games

Determination of One / All Nash Equilibria

We are interested in the computation, if possible in time polynomial in the representation of the game and / or the produced output, of the following problems:

- 2NASH:
  - INPUT: A bimatrix game with *non-negative rational* payoff matrices: $R, C \in \mathbb{Q}_{\geq 0}^{m \times n}$.
  - OUTPUT: Any (exact) Nash equilibrium, ie, any point of $NE(R, C)$.
- ALL2NASH:
  - INPUT: A bimatrix game with *non-negative rational* payoff matrices: $R, C \in \mathbb{Q}_{\geq 0}^{m \times n}$.
  - OUTPUT: All the extreme Nash equilibria, that determine $NE(R, C)$.

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We are interested in the computation, if possible in time polynomial in the representation of the game and / or the produced output, of the following problems:

- 2NASH:
  - INPUT: A bimatrix game with *non-negative rational* payoff matrices: $R, C \in \mathbb{Q}_{\geq 0}^{m \times n}$.
  - OUTPUT: Any (exact) Nash equilibrium, ie, any point of $NE(R, C)$.

Complexity of 2NASH

- [Nash (1950)] : *Existence* of NE points, for *finite* games.
- [Kuhn (1961), Mangasarian (1964), Lemke-Howson (1964), Rosenmüller (1971), Wilson (1971), Scarf (1967), Eaves (1972), Laan-Talman (1979), van den Elzen-Talman (1991), ...] : Algorithms for 2NASH. None of them is *provably* of polynomial complexity.
- [Savani-von Stengel (2004)] : LH may take an exponential number of pivots, to converge to a NE point, for *any* initial choice of the label to be rejected.
- [Goldberg-Papadimitriou-Savani (2011)] : Any algorithm for 2NASH based on the homotopy method, has no hope of being polynomial-time, (unless $\mathcal{P} = \mathcal{PSPACE}$).
- [CD / DGP (2006)] : $\mathcal{PPAD}-$completeness of kNASH, $\forall k \geq 2$.
- [Gilboa-Zemel (1989), Conitzer-Sandholm (2003)] : $\mathcal{NP}-$complete to determine *special* NE points (eg, $\exists$ more than one NE points? $\exists$NE with a lower bound on the payoff of one player? $\ldots$)

Formulations of 2NASH

Formulations for Nash Equilibrium Sets (I)

As a Linear Complementarity Problem (LCP)

- The set $NE(R, C)$ of Nash equilbria of the bimatrix game $\langle R, C \rangle$ can be expressed as the space of **non-zero feasible solutions** to a Linear Complementarity Problem, after proper *scaling* so that its points become probability-distribution pairs:

$$NE(R, C) \approx LCP(M, \mathbf{q}) - \{\mathbf{0}\}$$

where: $R, C \in \mathbb{R}^{m\times n}_{>0}$, $M = \begin{bmatrix} \mathbf{O} & -R \\ -C^T & \mathbf{O} \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} \mathbf{1}_m \\ \mathbf{1}_n \end{bmatrix}$, and

$$LCP(M, \mathbf{q}) = \{\ (\mathbf{w}, \mathbf{z}) :\ \mathbf{q} + M\mathbf{z} = \mathbf{w} \geq \mathbf{0};\ \mathbf{z} \geq \mathbf{0};\ \mathbf{w}^T\mathbf{z} = 0\ \}$$

Remark: The positivity of the payoff matrices is not a substantial constraint, due to invariance under shifting of $NE(R, C)$.

Formulations for Nash Equilibrium Sets (II)

As a Quadratic Programming Problem (QP)

- The space of Nash equilibria can be expressed as the space of **optimal solutions** in a Quadratic Programming Problem. Eg:

[Mangasarian-Stone (1964)]

$$(\text{MS}) \quad \begin{array}{rl} \text{minimize} & (r - \mathbf{x}^T R \mathbf{y}) + (c - \mathbf{x}^T C \mathbf{y}) \\ \text{s.t.} & r \cdot \mathbf{1} - R\mathbf{y} \geq 0 \\ & c \cdot \mathbf{1} - C^T \mathbf{x} \geq 0 \\ & \mathbf{x} \in \Delta_m \quad \mathbf{y} \in \Delta_n \end{array}$$

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\end{array}
$$

- Remark: The objective function is the sum of two *upper bounds* on two players' **regrets** against the given strategy of the opponent:

$$
\begin{array}{rrcl}
\text{ROW player:} & reg_I(\mathbf{x}, \mathbf{y}) & = & \max(R\mathbf{y}) - \mathbf{x}^T R\mathbf{y} \\
\text{column player:} & reg_{II}(\mathbf{x}, \mathbf{y}) & = & \max(C^T\mathbf{x}) - \mathbf{x}^T C\mathbf{y}
\end{array}
$$

Formulations of Nash Equilibrium Sets (III)

An alternative, *parameterized* QP formulation for $NE(R, C)$:

- **Marginal distributions** of a correlated strategy $\mathbf{W}$: $\forall k \in [m], \ \forall j \in [n]$,

$$x_k(\mathbf{W}) = \sum_{\ell \in [n]} W_{k,\ell}$$

$$y_j(\mathbf{W}) = \sum_{k \in [m]} W_{k,j}$$

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THEOREM: [Kontogiannis-Spirakis (2010)]

$NE(R,C)$ is the set of marginal distributions for all the **optimal solutions** in the following *parameterized* quadratic program, for *any* choice of the parameter $\lambda \in (0,1)$ and $Z(\lambda) = \lambda R + (1-\lambda)C$:

$$(\mathrm{KS}(\lambda)) \quad \begin{array}{rl} \text{minimize} & \sum_i \sum_j W_{i,j} Z(\lambda)_{i,j} - \mathbf{x}(\mathbf{W})^T Z(\lambda) \mathbf{y}(\mathbf{W}) \\ \text{s.t.} & \mathbf{W} \in CE(R,C) \end{array}$$

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Lemke & Howson Algorithm

A Combinatorial Algorithm for 2NASH (I)

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$$\bar{P} = \left\{(\mathbf{y}, u) : \mathbf{1}u - R\mathbf{y} \geq \mathbf{0};\ \ \mathbf{1}^T\mathbf{y} = 1;\ \ \mathbf{y} \geq \mathbf{0}\right\}$$
$$\bar{Q} = \left\{(\mathbf{x}, v) : \mathbf{1}v - C^T\mathbf{x} \geq \mathbf{0};\ \ \mathbf{1}^T\mathbf{x} = 1;\ \ \mathbf{x} \geq \mathbf{0}\right\}$$

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$$Q = \left\{\chi : \mathbf{1} - C^T\chi \geq \mathbf{0};\ \chi \geq \mathbf{0}\right\}$$

ASSUMPTION (wlog): The payoff matrices are positive.

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ASSUMPTION (wlog): The payoff matrices are positive.

- **Labels:** $\forall(\chi, \psi) \in Q \times P$,

$$L(\chi, \psi) = \begin{array}{c} \{i \in [m] : \chi_i = 0\} \ \cup \ (m + \{j \in [n] : (C^T\chi)_j = 1\}) \\ \bigcup \\ \{i \in [m] : (R\psi)_i = 1\} \ \cup \ (m + \{j \in [n] : \psi_j = 0\}) \end{array}$$

A Combinatorial Algorithm for 2NASH (II)

LH Algorithm [Lemke-Howson (1964)]

- Crucial Observation: A profile of strategies $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is Nash equilibrium iff the corresponding (non-zero) point $(\bar{\chi}, \bar{\psi}) \in Q \times P$ is **completely labeled**: all actions appear as labels in $L(\chi, \psi)$.
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Pseudocode of LH (for non-degenerate games)

1. Initialization: Starting from the completely labeled point $(\mathbf{0}, \mathbf{0})$ (artificial equilibrium), pivot-in (ie, label-out) an *arbitrary label* from $(\mathbf{0}, \mathbf{0})$.
2. Pivot-in (ie, label-out) the *unique double-label*, at the "other" polyhedron.
3. **if** the uniquely missing label is pivoted-out (labeled-in)
4. **then** return the new, *completely labeled* point (χ, ψ).
5. **else** goto step 2.

EXAMPLE: Execution of LH

	4	5	6
1	4 , 6	4 , 12	4 , 0
2	0 , 0	0 , 4	6 , 0
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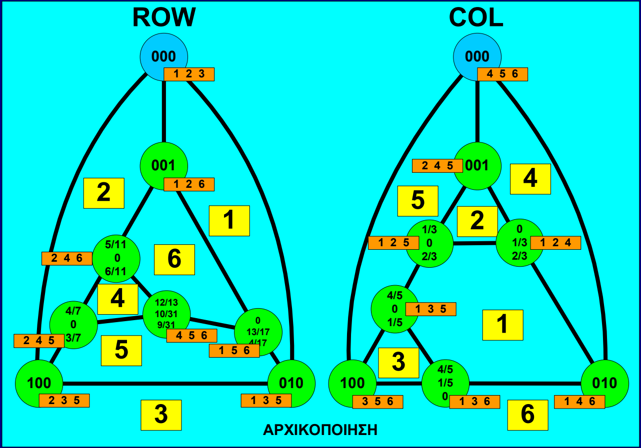

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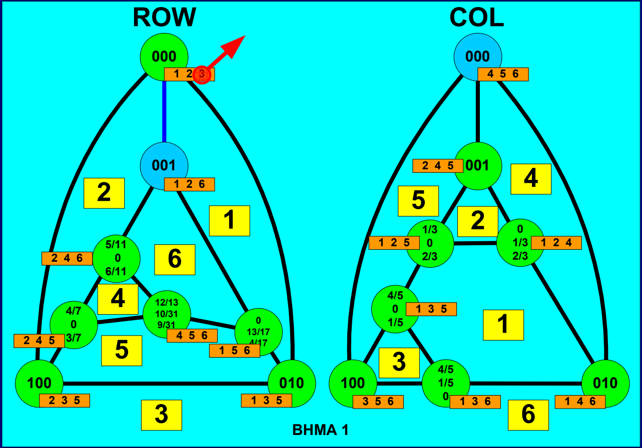

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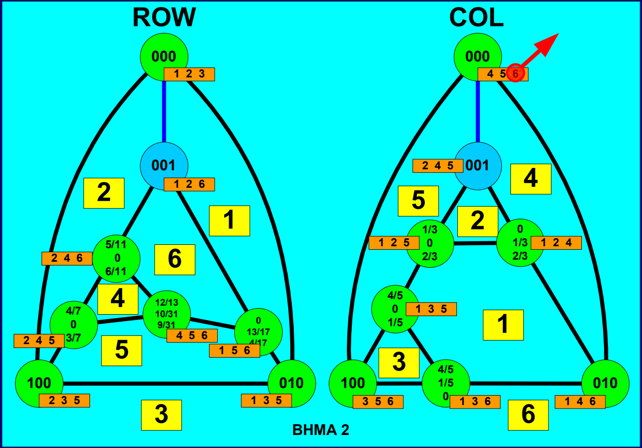

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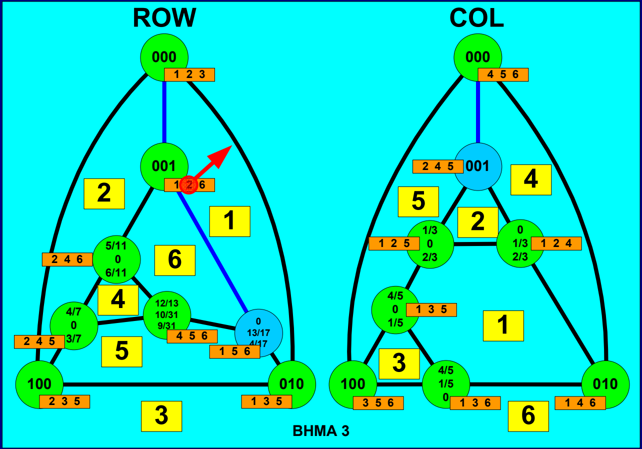

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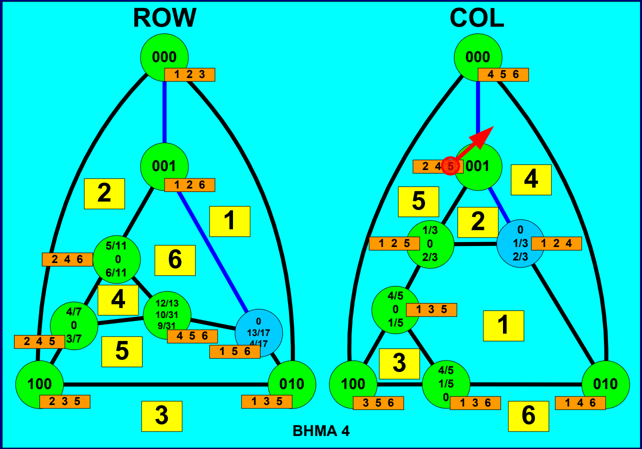

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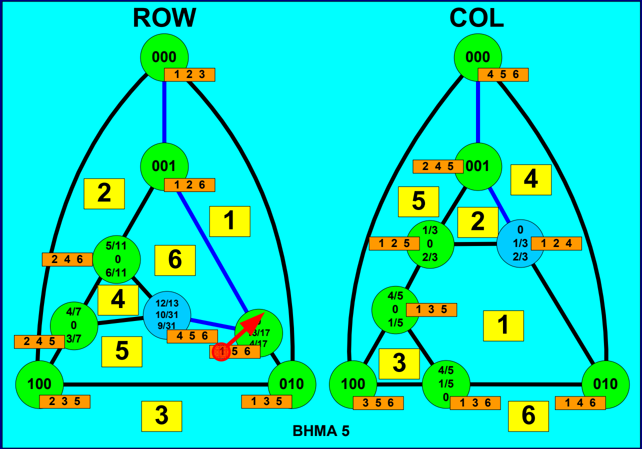

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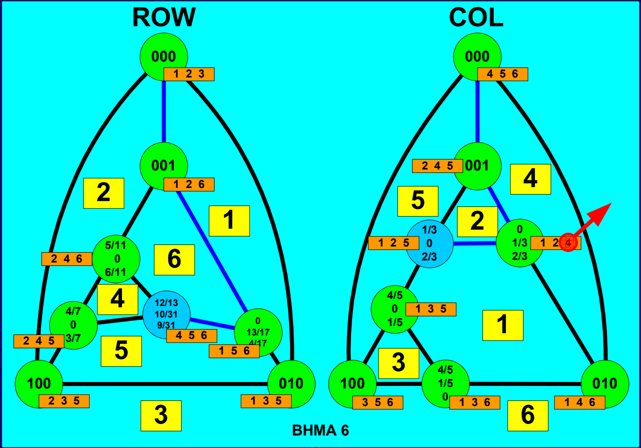

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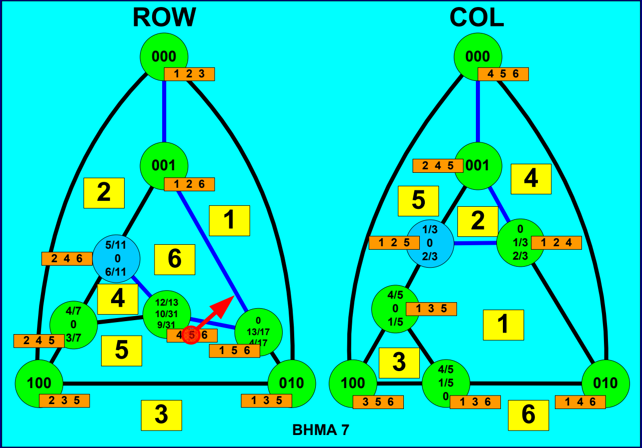

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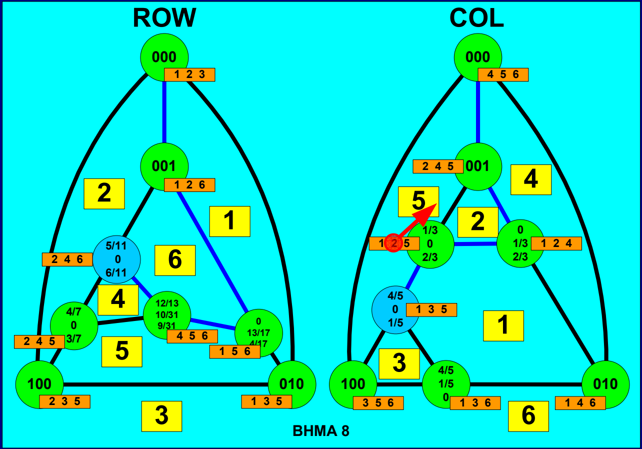

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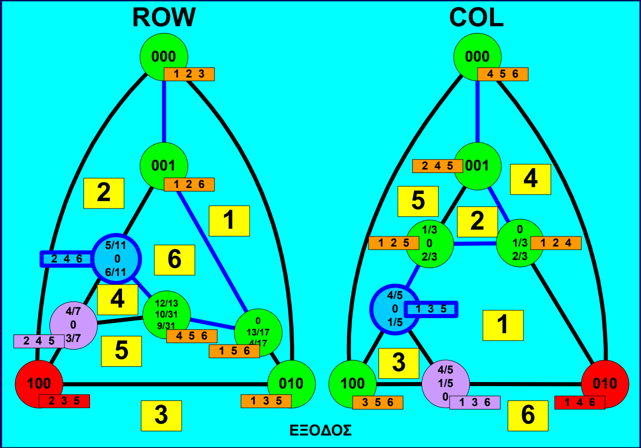


Why LH Works (for non-degenerate games)

- The artificial equilibrium $(\mathbf{0}, \mathbf{0})$ is *completely labeled*.

- The current pair of points, during the execution of LH, has *exactly one missing label* and *two adjacent edges* (one per polyhedron), for labeling-out each of the only two copies of the unique double-label.

- The "other ends" of these two edges lead to pairs of vertices with at most one double-label. The only missing label (if any) is always the one initially labeled-out.

- Each edge is traversed *towards new vertices*, so that no cycles may appear.

- Termination: Starting from one end of a *finite path*, we move towards the other end of this unique path, that is necessarily a completely-labeled pair of vertices.

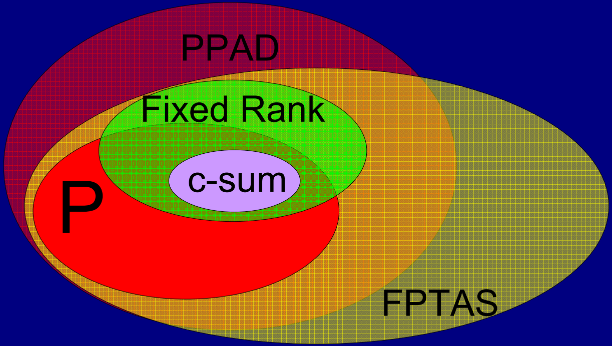
Skeleton of the Talk

1. Bimatrx Games Preliminaries
   - Complexity of 2NASH
   - Formulations for 2NASH
   - The algorithm of Lemke & Howson
2. Polynomial-time Tractable Subclasses
3. Approximability of 2NASH
   - Theoretical Analysis
   - Experimental Study
4. Conclusions

Polynomial-time Tractable Classes wrt 2NASH

- Zero-sum Games: Any profile of strategies is Nash equilibrium iff it is a MAXMIN profile
  - [J. von Neumann (1928)] : Existence proof for NE in zero-sum bimatrix games, based on a *Fixed Point* argument.
  - [Dantzig (1947)] : MAXMIN profiles for bimatrix games are equivalent to a pair of primal-dual linear programs.
  - [Khachiyan (1979), Karmakar (1984)] : Polynomial-tractability of LP.
- Constant-rank Games: $rank(R + C) = k$
  [Kannan-Theobald (2007)] : Existence of FPTAS for constructing ε-NE points.
- Rank-1 Games: $rank(R + C) = 1$
  [Adsul-Garg-Mehta-Sohoni (2011)] : Polynomial-time algorithm for determining *exact* Nash equilibria.
- (Very) Sparse Games & Games with Pure Equilibria: Relatively easy cases.

Other Tractable Classes?



- Constant-sum games are poly-time solvable. Constant-rank games admit a FPTAS.

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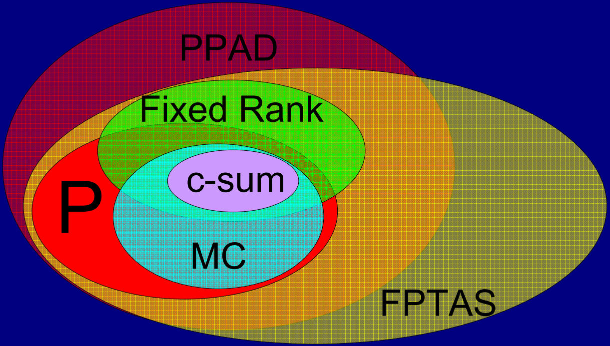

- Constant-sum games are poly-time solvable. Constant-rank games admit a FPTAS.

- Another subclass of poly-time solvable games: **Mutually-concave** games.

The Class of Mutually Concave Games

DEFINITION: Mutually Concave (MC) Games

A bimatrix game $\langle R, C \rangle$ is **mutually concave**, iff $\exists \lambda \in (0,1)$ s.t. for $Z(\lambda) = \lambda R + (1-\lambda) C$, the function $H_\lambda(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x}^T Z(\lambda) \mathbf{y}$ is *concave*.

Poly-time Solvability of 2NASH for MC Games

THEOREM: [Kontogiannis-Spirakis (2010)]

For arbitrary *normalized* bimatrix MC game $\langle R, C\rangle$, with *rational* payoff matrices, there exists a *(unique, except for some trivial cases)* rational number $\lambda^* \in (0,1)$ s.t. for $Z(\lambda) = \lambda^* R + (1-\lambda^*)C$ the following quadratic program is convex, and therefore polynomial-time solvable:

$$\begin{array}{rl} \text{minimize} & \sum_i \sum_j W_{i,j} Z(\lambda^*)_{i,j} - \mathbf{x}(\mathbf{W})^T Z(\lambda^*) \mathbf{y}(\mathbf{W}) \\ \text{s.t.} & \mathbf{W} \in CE(R,C) \end{array}$$

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$$\begin{array}{rl} \text{minimize} & \lambda^* \cdot (r - \mathbf{x}^T R\mathbf{y}) + (1-\lambda^*)\cdot(c - \mathbf{x}^T C\mathbf{y}) \\ \text{s.t.} & r\cdot\mathbf{1} - R\mathbf{y} \geq 0 \\ & c\cdot\mathbf{1} - C^T\mathbf{x} \geq 0 \\ & \mathbf{x}\in\Delta_m \qquad \mathbf{y}\in\Delta_n \end{array}$$

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For arbitrary *normalized* bimatrix MC game $\langle R, C\rangle$, with *rational* payoff matrices, there exists a *(unique, except for some trivial cases)* rational number $\lambda^* \in (0,1)$ s.t. for $Z(\lambda) = \lambda^* R + (1-\lambda^*)C$ the following quadratic program is convex, and therefore polynomial-time solvable:

$$\begin{array}{rl} \text{minimize} & \lambda^* \cdot (r - \mathbf{x}^T R\mathbf{y}) + (1-\lambda^*)\cdot(c - \mathbf{x}^T C\mathbf{y}) \\ \text{s.t.} & r\cdot \mathbf{1} - R\mathbf{y} \geq 0 \\ & c\cdot \mathbf{1} - C^T\mathbf{x} \geq 0 \\ & \mathbf{x}\in\Delta_m \qquad \mathbf{y}\in\Delta_n \end{array}$$

- Are we done?

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- Are we done?

NOT YET!!!

It is crucial to be able to **recognize in polynomial time** whether a bimatrix game belongs to the MC class.

Recognition of Poly-time Solvable Bimatrix Classes

- Trivial issue for a game $\langle R, C\rangle$ that...
  - ...has *constant-sum payoffs*.
  - ...has *constant rank*.
  - ...possesses a *pure Nash* equilibrium.
  - ...is a *very sparse win-lose* game.

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- Is there a way to detected *mutual concavity* also in polynomial time?

☺ **YES!!!**

Characterizations of Mutual Concavity

PROPOSITION: [Kontogiannis-Spirakis (2010)]

A bimatrix game $\langle R, C\rangle$ is mutually concave iff any of the following properties holds:

1. $\exists\lambda \in (0,1) : \xi^T Z(\lambda)\psi = 0$, for any pair of *directions of change in strategies* ξ, ψ for the two players (ie, such that $\mathbf{1}^T\xi = \mathbf{1}^T\psi = 0$).
2. $\exists\lambda \in (0,1) : Z(\lambda)\cdot\psi = \mathbf{1}\cdot c$, for an arbitrary constant c and any direction of change, $\psi \in \mathbb{R}^n : \mathbf{1}^T\psi = 0$, for the strategy of the column player.

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COROLLARY: Constant-Sum Games & MC Class

Every constant-sum game $\langle A, -A + c\cdot\mathbf{1}\cdot\mathbf{1}^T\rangle$ is mutually concave.

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2. $\exists \lambda \in (0,1) : Z(\lambda) \cdot \psi = \mathbf{1} \cdot c$, for an arbitrary constant c and any direction of change, $\psi \in \mathbb{R}^n : \mathbf{1}^T \psi = 0$, for the strategy of the column player.

COROLLARY: Constant-Sum Games & MC Class

Every constant-sum game $\langle A, -A + c \cdot \mathbf{1} \cdot \mathbf{1}^T \rangle$ is mutually concave.

- Question: How do we detect the existence (or not) of the proper λ^*-value?

Mutual Concavity for 2×2 Games

PROPOSITION: [Kontogiannis-Spirakis (2010)]

For any 2×2 game $\langle A, B \rangle$, let $\alpha = A_{1,1} + A_{2,2} - A_{1,2} - A_{2,1}$ and $b = B_{1,1} + B_{2,2} - B_{1,2} - B_{2,1}$. Then, $\langle A, B \rangle$ is mutually concave iff:

$$\boxed{\alpha = b = 0 \vee \min\{\alpha, b\} < 0 < \max\{\alpha, b\}}$$

If $\alpha, b \neq 0$, then the *unique value* $\lambda^* = \frac{-b}{\alpha - b}$ proves (if it holds) the mutual concavity of the game.

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If $\alpha, b \neq 0$, then the *unique value* $\lambda^* = \frac{-b}{\alpha - b}$ proves (if it holds) the mutual concavity of the game.

COROLLARY: A Necessary Condition for MC

If an $m \times n$ bimatrix game $\langle A, B \rangle$ is mutually concave, then the following must hold: $\exists \lambda^* \in (0,1) : \forall 1 \leq i < k \leq m, \forall 1 \leq j < \ell \leq n,$

$$[\ \alpha_{ik,j\ell} = b_{ik,j\ell} = 0\ ]$$
$$\vee$$
$$\left[\ \max\{\alpha_{ik,j\ell}, b_{ik,j\ell}\} > 0 > \min\{\alpha_{ik,j\ell}, b_{ik,j\ell}\} \quad \wedge \quad \lambda^* = \frac{-b_{ik,j\ell}}{\alpha_{ik,j\ell} - b_{ik,j\ell}}\ \right]$$

Examples of MC / non-MC Games

MC-games

ROW \ col	left	right
TOP	2 , 1	1,1+γ
BOTTOM	1 , 1	3 , 0

(a) MC-game, $\forall \gamma > -1$.

PRISONER 1 \ prisoner 2	betray	silent
BETRAY	4 , 4	10 , 0
SILENT	0 , 10	6 , 6

(b) MC version of PD game

ROW \ col	left	right
TOP	2 , 1	1,1+γ
MIDDLE	1 , 1	3 , 0
BOTTOM	δ_1 , δ_2	$\varepsilon_1 , \varepsilon_2$

(c) MC-game $\forall \gamma > -1$, arbitrary $\delta_1, \delta_2, \varepsilon_1$ and ε_2 be their function.

Non-MC-games

SHE \ he	theater? ok	basket !!!
THEATER !!!	5 , 1	0 , 0
BASKET? OK	0 , 0	1 , 5

(d) Battle of Sexes.

PRISONER 1 \ prisoner 2	betray	silent
BETRAY	-5 , -5	0 , -10
SILENT	-10 , 0	-1 , -1

(e) Non-MC version of PD game.

ROW \ col	dare	chicken
DARE	0 , 0	7 , 2
CHIKEN	2 , 7	6 , 6

(f) Chicken Game.

Checking Mutual Concavity in Poly-time

ALGORITHM: Detecting MC for non-trivial games

INPUT: $(R, C) \in \mathbb{R}^{m \times n}$.

(1) Check all 2×2 subgames of (R, C), for the induced $\lambda-$values.

(2) **if** all 2×2 subgames have zero $\alpha-$ and $b-$values

(3) **then return ("there is PNE")**

(4) **if** there is a *unique* induced λ^*-value by all 2×2 subgames with nonzero $\alpha-$ and $b-$values

(5) **then if** $\begin{pmatrix} \mathbf{0} & Z(\lambda^*) \\ Z(\lambda^*)^T & \mathbf{0} \end{pmatrix}$ is negative semidefinite

(6) **then return ("MC-game")**

(7) **return ("non-MC-game")**

MC-games vs. Fixed-Rank Games

- Even rank-1 games may not be MC-games.

		prisoner 2	
		betray	silent
PRISONER 1	BETRAY	-5 , -5	0 , -10
	SILENT	-10 , 0	-1 , -1

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BETRAY	-5 , -5	0 , -10
SILENT	-10 , 0	-1 , -1

- There exist MC games that have full rank. Eg, for:

$$Z = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 & 32 & 64 \\ 0 & 1 & 3 & 7 & 15 & 31 & 63 \\ 2 & 3 & 5 & 9 & 17 & 33 & 65 \\ -1 & 0 & 2 & 6 & 14 & 30 & 62 \\ 3 & 4 & 6 & 10 & 18 & 34 & 66 \\ -2 & -1 & 1 & 5 & 13 & 29 & 61 \\ 4 & 5 & 7 & 11 & 19 & 35 & 67 \end{bmatrix}$$

with $rank(Z) = 2$, the game $\langle R, C \rangle$ with $R = I_7$ and $C = \frac{4}{3}Z - \frac{1}{3}R$ is indeed (cf. next slide's characterization) an MC-game, but it has $rank(R + C) = 7$.

MC Games vs. Strategically Zero-Sum Games

PROPOSITION: [Kontogiannis-Spirakis (2010)]

For any $m, n \geq 2$ and payoff matrices $R, C \in \mathbb{R}^{m\times n}$, the game $\langle R, C\rangle$ is an MC-game iff: $\exists\lambda \in (0,1), \exists\mathbf{a} \in \mathbb{R}^m, \exists\mathbf{d} = [0, d_2, \ldots, d_n]^T \in \mathbb{R}^n$:

$$\forall j \in [n],\ \ Z(\lambda)[*, j] = -d_j \cdot \mathbf{1} + \mathbf{a}$$

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Some Remarks:

1. The MC-class *matches* the class of strategically-zero-sum (SZS) games of [Moulin-Vial (1978)] .
2. Characterizing the SZS-property implies the solution of a large *linear program*, in time $O\left(n^6\right)$.
3. Characterizing the MC-property implies the solution of a much smaller *quadratic program*, in time $O\left(n^4\right)$.

Skeleton of the Talk

1. Bimatrx Games Preliminaries
   - Complexity of 2NASH
   - Formulations for 2NASH
   - The algorithm of Lemke & Howson
2. Polynomial-time Tractable Subclasses
3. Approximability of 2NASH
   - Theoretical Analysis
   - Experimental Study
4. Conclusions

Reminder...

DEFINITION: Approximate Nash Equilibria

For a *normalized* bimatrix game $\langle R, C\rangle$, a profile of *(uncorrelated) strategies* $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is:

- $\varepsilon-$**approximate Nash equilibrium** (ε-NE) iff no player can improve her expected payoff more than an additive term of $\varepsilon \geq 0$ by *unilaterally changing* her strategy, against the given strategy of the opponent: $\forall \mathbf{x} \in \Delta_m$, $\forall \mathbf{y} \in \Delta_n$,
$$\bar{\mathbf{x}}^T R \bar{\mathbf{y}} \geq \mathbf{x}^T R \bar{\mathbf{y}} - \varepsilon \ \wedge \ \bar{\mathbf{x}}^T C \bar{\mathbf{y}} \geq \bar{\mathbf{x}}^T C \mathbf{y} - \varepsilon$$
- $\varepsilon-$**well supported approximate Nash equilibrium** (ε-WSNE) iff no player assigns positive probability mass to actions that are less than an additive term $\varepsilon \geq 0$ than the maximum payoff she may get against the given strategy of the opponent: $\forall i \in [m]$, $\forall j \in [n]$,

$$[\bar{x}_i > 0 \ \rightarrow \ \mathbf{e_i}^T R \bar{\mathbf{y}} \geq \max(R\bar{\mathbf{y}}) - \varepsilon] \ \wedge \ [\bar{y}_j > 0 \rightarrow \bar{\mathbf{x}}^T C \mathbf{e_j} \geq \max(\bar{\mathbf{x}}^T C) - \varepsilon]$$

Approximability of 2NASH

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ε-NE:

- [Kontogiannis-Panagopoulou-Spirakis (2006)] : 0.75
- [Daskalakis-Papadimitriou-Mehta (2006)] : 0.5
- [Daskalakis-Papadimitriou-Mehta (2007)] : ~ 0.38
- [Bosse-Byrka-Markakis (2007)] : ~ 0.36
- [Spirakis-Tsaknakis (2007)] : ~ 0.3393
- [Kontogiannis-Spirakis (2011)] : $\sim 1/3 + \delta$, for any *constant* $\delta > 0$ and *symmetric games*.

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- [Kontogiannis-Spirakis (2007)] : 0.667
- [Fearnley-Goldberg-Savani-Sørensen (2012)] : <0.667

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- Question: Can we break any of the bounds, $1/3$ for ε-NE and $2/3$ for ε-WSNE? Is there a PTAS for either case?

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Theoretical Analysis of SYMMETRIC-2NASH Approximations

About Symmetric Bimatrix Games...

- $n \times n$ games $\langle R, C \rangle$, where the players may exchange roles (ie, the payoff matrices are: $R = S,\ C = S^T$.
- [Nash (1950)] : Every *finite symmetric* game has a **symmetric Nash equilibrium** (in which all players adopt the same strategy).
- For *symmetric strategy profiles* in symmetric bimatrix games, the players have common expected payoffs and also common expected payoff vectors, against the opponent's strategy.
- A *formalism* for SYMMETRIC-2NASH: For

$$\mathbf{x} = \binom{\mathbf{z}}{\mathbf{z}},\ Q = \begin{pmatrix} \mathbf{O} & S^T \\ S & \mathbf{O} \end{pmatrix},$$

$$\text{(SMS)} \quad \begin{array}{|lrcrcl|} \text{minimize} & \multicolumn{5}{l|}{f(s,\mathbf{z}) = s - \mathbf{z}^T S\mathbf{z} = s - \frac{1}{2}\mathbf{x}^T Q\mathbf{x}} \\ \text{s.t.:} & -\mathbf{1}s & + & S\mathbf{z} & & \leq \mathbf{0} \\ & & - & \mathbf{1}^T\mathbf{z} & +1 & = 0 \\ & s \in \mathbb{R}, & & \mathbf{z} \in \mathbb{R}^n_{\geq 0} & & \end{array}$$

Necessary Optimality (KKT) Conditions for (SMS)

(KKTSMS)

$$\nabla f(\bar{s}, \bar{\mathbf{z}}) = \begin{pmatrix} 1 \\ -S\bar{\mathbf{z}} - S^T\bar{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \mathbf{1}^T\bar{\mathbf{w}} \\ -S^T\bar{\mathbf{w}} + \bar{\mathbf{u}} + \mathbf{1}\bar{\zeta} \end{pmatrix}$$

$$0 \leq \begin{pmatrix} \bar{\mathbf{w}} \\ \bar{\mathbf{u}} \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{1}\bar{s} - S\bar{\mathbf{z}} \\ \bar{\mathbf{z}} \end{pmatrix} \leq \delta$$

$$\bar{s} \in \mathbb{R}, \quad S\bar{\mathbf{z}} \leq \mathbf{1}\bar{s}, \quad \mathbf{1}^T\bar{\mathbf{z}} = 1, \quad \bar{\mathbf{z}} \geq \mathbf{0}$$

$$\bar{\mathbf{w}} \geq \mathbf{0}, \quad \bar{\zeta} \in \mathbb{R}, \quad \bar{\mathbf{u}} \geq \mathbf{0}$$

- **δ-KKT Points** of (SMS): Feasible solutions $(\bar{s}, \bar{\mathbf{z}}, \bar{\mathbf{w}}, \bar{\zeta}, \bar{\mathbf{u}})$ for (KKTSMS).
- Some Remarks:
  1. The Lagrange multiplier $\bar{\mathbf{w}}$ is an alternative strategy for the players, ie, a point from $\Delta([n])$.
  2. $\bar{\mathbf{w}}$ is also a *$\delta-$approximate best response* of the ROW player against the given strategy $\bar{\mathbf{z}}$ of the opponent: $\bar{s} \leq \bar{\mathbf{w}}^T S\bar{\mathbf{z}} + \delta$.

Computing (approximate) KKT Points of QPs

- Exact computation of a KKT point of Quadratic Programs: $\mathcal{NP}$-hard problem.

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THEOREM: Approximate KKT Points in QP [Ye (1998)]

There is a FPTAS for computing $\delta-$KKT points of a n-varialbe Quadratic Program, in time:

$$O\left(\left[\frac{n^6}{\delta}\log\left(\frac{1}{\delta}\right)+n^4\log(n)\right]\cdot\left[\log\log\left(\frac{1}{\delta}\right)+\log(n)\right]\right)$$

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- Question: What is the *quality as a Nash approximation* of a $\delta-$KKT point?

A Fundamental Property of (KKTSMS) – (I)

LEMMA: [Kontogiannis-Spirakis (SEA 2011)]

For any $m, n \geq 2$, $S \in [0,1]^{m \times n}$, and any point $(\max(S\bar{\mathbf{z}}), \bar{\mathbf{z}}, \bar{\mathbf{w}}, \bar{\mathbf{u}}, \bar{\zeta}) \in (KKTSMS)$ the following properties hold:

1. $\bar{\zeta} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S \bar{\mathbf{z}}$.
2. $2f(\bar{\mathbf{z}}) = \bar{\mathbf{w}}^T S \bar{\mathbf{w}} - \bar{\mathbf{z}}^T S \bar{\mathbf{w}} - \bar{\mathbf{w}}^T \bar{\mathbf{u}}$.
3. $2f(\bar{\mathbf{z}}) + f(\bar{\mathbf{w}}) = R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}^T \bar{\mathbf{u}}$.

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3. $2f(\bar{\mathbf{z}}) + f(\bar{\mathbf{w}}) = R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}^T \bar{\mathbf{u}}$.

Some Remarks:

- $f(\bar{\mathbf{z}}) = \max(S\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S \bar{\mathbf{z}}$ is either player's regret, for the *symmetric* profile $(\bar{\mathbf{z}}, \bar{\mathbf{z}})$.
- $R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) = \max(S\bar{\mathbf{w}}) - \bar{\mathbf{z}}^T S \bar{\mathbf{w}} \leq 1$ is (only) the row player's regret, for the *asymmetric profile* $(\bar{\mathbf{w}}, \bar{\mathbf{z}})$.
- The third property assures that any (exact) KKT point (is not necessarily itself, but) indicates a $1/3-$NE of $\langle S, S^T \rangle$, in normalized games.

A Fundamental Property of (KKTSMS) – (II)

Proof of the Lemma

$$-S\bar{\mathbf{z}} - S^T\bar{\mathbf{z}} = -S^T\bar{\mathbf{w}} + \bar{\mathbf{u}} + \mathbf{1}\bar{\zeta}$$

$$\Rightarrow \quad \begin{cases} -\bar{\mathbf{z}}^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S^T\bar{\mathbf{z}} = -\bar{\mathbf{z}}^T S^T\bar{\mathbf{w}} + \underbrace{\bar{\mathbf{z}}^T\bar{\mathbf{u}}}_{=0} + \underbrace{\bar{\mathbf{z}}^T\mathbf{1}}_{=1}\bar{\zeta} \\ -\bar{\mathbf{w}}^T S\bar{\mathbf{z}} - \bar{\mathbf{w}}^T S^T\bar{\mathbf{z}} = -\bar{\mathbf{w}}^T S^T\bar{\mathbf{w}} + \bar{\mathbf{w}}^T\bar{\mathbf{u}} + \underbrace{\bar{\mathbf{w}}^T\mathbf{1}}_{=1}\bar{\zeta} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \bar{\zeta} = -2\bar{\mathbf{z}}^T S\bar{\mathbf{z}} + \bar{\mathbf{z}}^T S^T\bar{\mathbf{w}} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \\ \bar{\zeta} = -\bar{\mathbf{w}}^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}} + \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{w}}^T\bar{\mathbf{u}} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \bar{\zeta} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \\ 2f(\bar{\mathbf{z}}) = -2\bar{\mathbf{z}}^T S\bar{\mathbf{z}} + 2\bar{\mathbf{w}}^T S\bar{\mathbf{z}} = -\bar{\mathbf{z}}^T S\bar{\mathbf{w}} + \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{w}}^T\bar{\mathbf{u}} \end{cases}$$

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$$-S\bar{\mathbf{z}} - S^T\bar{\mathbf{z}} = -S^T\bar{\mathbf{w}} + \bar{\mathbf{u}} + \mathbf{1}\bar{\zeta}$$

$$\Rightarrow \begin{cases} -\bar{\mathbf{z}}^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S^T\bar{\mathbf{z}} = -\bar{\mathbf{z}}^T S^T\bar{\mathbf{w}} + \underbrace{\bar{\mathbf{z}}^T\bar{\mathbf{u}}}_{=0} + \underbrace{\bar{\mathbf{z}}^T\mathbf{1}}_{=1}\bar{\zeta} \\ -\bar{\mathbf{w}}^T S\bar{\mathbf{z}} - \bar{\mathbf{w}}^T S^T\bar{\mathbf{z}} = -\bar{\mathbf{w}}^T S^T\bar{\mathbf{w}} + \bar{\mathbf{w}}^T\bar{\mathbf{u}} + \underbrace{\bar{\mathbf{w}}^T\mathbf{1}}_{=1}\bar{\zeta} \end{cases}$$

$$\Rightarrow \begin{cases} \bar{\zeta} = -2\bar{\mathbf{z}}^T S\bar{\mathbf{z}} + \bar{\mathbf{z}}^T S^T\bar{\mathbf{w}} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \\ \bar{\zeta} = -\bar{\mathbf{w}}^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}} + \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{w}}^T\bar{\mathbf{u}} \end{cases}$$

$$\Rightarrow \begin{cases} \bar{\zeta} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \\ 2f(\bar{\mathbf{z}}) = -2\bar{\mathbf{z}}^T S\bar{\mathbf{z}} + 2\bar{\mathbf{w}}^T S\bar{\mathbf{z}} = -\bar{\mathbf{z}}^T S\bar{\mathbf{w}} + \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{w}}^T\bar{\mathbf{u}} \end{cases}$$

- We add $f(\bar{\mathbf{w}}) = \max(S\bar{\mathbf{w}}) - \bar{\mathbf{w}}^T S\bar{\mathbf{w}}$ to both sides of the equation:

$$2f(\bar{\mathbf{z}}) + f(\bar{\mathbf{w}}) = \max(S\bar{\mathbf{w}}) - \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}} + \bar{\mathbf{w}}^T S\bar{\mathbf{w}} - \bar{\mathbf{w}}^T\bar{\mathbf{u}}$$

$$\Rightarrow \boxed{3 \cdot \min\{f(\bar{\mathbf{z}}), f(\bar{\mathbf{w}})\} \leq 2f(\bar{\mathbf{z}}) + f(\bar{\mathbf{w}}) = R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}^T\bar{\mathbf{u}} \leq 1}$$

$(<1/3)$–NE from a Given Exact KKT Point – (I)

THEOREM: [Kontogiannis-Spirakis (2011)]

Starting from any (exact) KKT point of (KKTSMS) for a normalized symmetric bimatrix game $\langle S, S^T \rangle$, computing a $\left(< \frac{1}{3}\right)$–NE can be done in polynomial time.

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Proof Sketch:

- $(\max(S\bar{\mathbf{z}}), \bar{\mathbf{z}}, \bar{\mathbf{w}}, \bar{\mathbf{u}}, \bar{\zeta})$: The given KKT point, along with the proper Lagrange multipliers.
- **if** $f(\bar{\mathbf{z}}) \neq f(\bar{\mathbf{w}})$ **then** $3 \cdot \min\{f(\bar{\mathbf{z}}), f(\bar{\mathbf{w}})\} < R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}^T \bar{\mathbf{u}} \leq 1$.

$\therefore$ ASSUMPTION 1: $f(\bar{\mathbf{z}}) = f(\bar{\mathbf{w}}) = \frac{1}{3}$.

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$\therefore$ ASSUMPTION 1: $f(\bar{\mathbf{z}}) = f(\bar{\mathbf{w}}) = \frac{1}{3}$.

- **if** $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \notin$ (KKTSMS)
  then starting from $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}})$, the next step towards a KKT point will give a $(<1/3)$–NE.

$\therefore$ ASSUMPTION 2: $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in$ (KKTSMS).

$(<1/3)$–NE from a Given Exact KKT Point – (II)

- $(\bar{\mathbf{w}}', \bar{\mathbf{u}}', \bar{\zeta}')$: The appropriate Lagrange multipliers for $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in (\text{KKTSMS})$.

- From the **Basic Lemma**, applied now to $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}})$:

$$2f(\bar{\mathbf{w}}) + f(\bar{\mathbf{w}}') = R_I(\bar{\mathbf{w}}, \bar{\mathbf{w}}') - (\bar{\mathbf{w}}')^T \bar{\mathbf{u}}'$$

$(<1/3)$–NE from a Given Exact KKT Point – (II)

- $(\bar{\mathbf{w}}', \bar{\mathbf{u}}', \bar{\zeta}')$: The appropriate Lagrange multipliers for $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in$ (KKTSMS).
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- Observation 1:

$$\begin{aligned} & 1 = 3f(\bar{\mathbf{z}}) = R_I(\bar{\mathbf{z}}, \bar{\mathbf{w}}) - \bar{\mathbf{w}}^T \bar{\mathbf{u}} \\ \Rightarrow \quad & \max(S\bar{\mathbf{w}}) = 1 \;\wedge\; \bar{\mathbf{z}}^T S\bar{\mathbf{w}} = 0 \;\wedge\; \bar{\mathbf{w}}^T \bar{\mathbf{u}} = 0 \\ & 1 = 3f(\bar{\mathbf{w}}) = R_I(\bar{\mathbf{w}}, \bar{\mathbf{w}}') - (\bar{\mathbf{w}}')^T \bar{\mathbf{u}}' \\ \Rightarrow \quad & \boxed{\max(S\bar{\mathbf{w}}') = 1 \;\wedge\; \bar{\mathbf{w}}^T S\bar{\mathbf{w}}' = 0 \;\wedge\; (\bar{\mathbf{w}}')^T \bar{\mathbf{u}}' = 0} \end{aligned}$$

$(<1/3)$–NE from a Given Exact KKT Point – (II)

- $(\bar{\mathbf{w}}', \bar{\mathbf{u}}', \bar{\zeta}')$: The appropriate Lagrange multipliers for $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in$ (KKTSMS).
- From the **Basic Lemma**, applied now to $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}})$:

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- Observation 2: $\frac{1}{3} = f(\bar{\mathbf{w}}) = \max(S\bar{\mathbf{w}}) - \bar{\mathbf{w}}^T S\bar{\mathbf{w}} \Rightarrow \boxed{\bar{\mathbf{w}}^T S\bar{\mathbf{w}} = \frac{2}{3}}$

(< 1/3)–NE from a Given Exact KKT Point – (III)

- **if** $f(\bar{\mathbf{w}}) \neq f(\bar{\mathbf{w}}')$ **then** $3\min\{f(\bar{\mathbf{w}}), f(\bar{\mathbf{w}}')\} < 1$.

$\therefore$ ASSUMPTION 3: $f(\bar{\mathbf{z}}) = f(\bar{\mathbf{w}}) = f(\bar{\mathbf{w}}') = \frac{1}{3}$.

$(<1/3)$–NE from a Given Exact KKT Point – (III)

- **if** $f(\bar{\mathbf{w}}) \neq f(\bar{\mathbf{w}}')$ **then** $3\min\{f(\bar{\mathbf{w}}), f(\bar{\mathbf{w}}')\} < 1$.

$\therefore$ ASSUMPTION 3: $f(\bar{\mathbf{z}}) = f(\bar{\mathbf{w}}) = f(\bar{\mathbf{w}}') = \frac{1}{3}$.

- $(\max(S\bar{\mathbf{z}}), \bar{\mathbf{z}}) \in$ (KKTSMS):

$$\left.\begin{array}{r} -S\bar{\mathbf{z}} - S^T\bar{\mathbf{z}} + S^T\bar{\mathbf{w}} = \bar{\mathbf{u}} + \mathbf{1}\bar{\zeta} \\ \bar{\zeta} = f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \end{array}\right\} \Rightarrow$$

$$-(\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}}' + \underbrace{\bar{\mathbf{w}}^T S\bar{\mathbf{w}}'}_{=0} = \bar{\mathbf{w}}'^T\bar{\mathbf{u}} + f(\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \Rightarrow$$

$$0 \leq (\bar{\mathbf{w}}')^T\bar{\mathbf{u}} = -(\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}}' - f(\bar{\mathbf{z}}) + \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \Rightarrow$$

$$\boxed{(\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \leq -\frac{1}{3} - \bar{\mathbf{z}}^T S\bar{\mathbf{w}}' \leq -\frac{1}{3} < 0}$$

$(<1/3)$–NE from a Given Exact KKT Point – (IV)

- $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in$ (KKTSMS):

$$\left.\begin{array}{r} -S\bar{\mathbf{w}} - S^T\bar{\mathbf{w}} + S^T\bar{\mathbf{w}}' = \bar{\mathbf{w}}' + \mathbf{1}\bar{\zeta}' \\ \bar{\zeta}' = f(\bar{\mathbf{w}}) - \bar{\mathbf{w}}^T S\bar{\mathbf{w}} = -\frac{1}{3} \end{array}\right\} \quad \Rightarrow$$

$$-\underbrace{\bar{\mathbf{z}}^T S\bar{\mathbf{w}}}_{=0} - \underbrace{\bar{\mathbf{z}}^T S^T\bar{\mathbf{w}}}_{=\max(S\bar{\mathbf{z}})} + \bar{\mathbf{z}}^T S^T\bar{\mathbf{w}}' = \bar{\mathbf{z}}^T\bar{\mathbf{u}}' - \frac{1}{3} \quad \Rightarrow$$

$$0 \le \bar{\mathbf{z}}^T\bar{\mathbf{u}}' = \frac{1}{3} - \max(S\bar{\mathbf{z}}) + (\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} \quad \Rightarrow$$

$$\underbrace{\max(S\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}}}_{=f(\bar{\mathbf{z}})=\frac{1}{3}} \le \frac{1}{3} + (\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \quad \Rightarrow$$

$$\boxed{(\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \ge 0}$$

(< 1/3)–NE from a Given Exact KKT Point – (IV)

- $(\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in (\text{KKTSMS})$:

$$\left.\begin{array}{r} -S\bar{\mathbf{w}} - S^T\bar{\mathbf{w}} + S^T\bar{\mathbf{w}}' = \bar{\mathbf{w}}' + \mathbf{1}\bar{\zeta}' \\ \bar{\zeta}' = f(\bar{\mathbf{w}}) - \bar{\mathbf{w}}^T S\bar{\mathbf{w}} = -\frac{1}{3} \end{array}\right\} \Rightarrow$$

$$-\underbrace{\bar{\mathbf{z}}^T S\bar{\mathbf{w}}}_{=0} - \underbrace{\bar{\mathbf{z}}^T S^T\bar{\mathbf{w}}}_{=\max(S\bar{\mathbf{z}})} + \bar{\mathbf{z}}^T S^T\bar{\mathbf{w}}' = \bar{\mathbf{z}}^T\bar{\mathbf{u}}' - \frac{1}{3} \Rightarrow$$

$$0 \leq \bar{\mathbf{z}}^T\bar{\mathbf{u}}' = \frac{1}{3} - \max(S\bar{\mathbf{z}}) + (\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} \Rightarrow$$

$$\underbrace{\max(S\bar{\mathbf{z}}) - \bar{\mathbf{z}}^T S\bar{\mathbf{z}}}_{=f(\bar{\mathbf{z}})=\frac{1}{3}} \leq \frac{1}{3} + (\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \Rightarrow$$

$$\boxed{(\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \geq 0}$$

$\therefore$ **if** $f(\bar{\mathbf{z}}) = f(\bar{\mathbf{w}}) = f(\bar{\mathbf{w}}') = \frac{1}{3} \;\wedge\; (\max(S\bar{\mathbf{z}}), \bar{\mathbf{z}}), (\max(S\bar{\mathbf{w}}), \bar{\mathbf{w}}) \in (\text{KKTSMS})$

then $\boxed{0 \leq (\bar{\mathbf{w}}')^T S\bar{\mathbf{z}} - \bar{\mathbf{z}}^T S\bar{\mathbf{z}} \leq -\frac{1}{3}}$ /* CONTRADICTION */

Efficient Computation of $\left(\frac{1}{3}+\delta\right)-$NE

THEOREM: [Kontogiannis-Spirakis (2011)]

For any normalized symmetric bimatrix game $\langle S, S^T\rangle$ with *rational* payoff values, $S \in [0,1]^{n\times n}$, and any *constant* $\delta > 0$, it is possible to construct a *symmetric* $(1/3+\delta)-$NE point, in time polynomial in the description of the game and quasi-linear in the value of δ.

Efficient Computation of $\left(\frac{1}{3}+\delta\right)-$NE

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For any normalized symmetric bimatrix game $\langle S, S^T \rangle$ with *rational* payoff values, $S \in [0,1]^{n \times n}$, and any *constant* $\delta > 0$, it is possible to construct a *symmetric* $(1/3+\delta)-$NE point, in time polynomial in the description of the game and quasi-linear in the value of δ.

- Similar proof with that for the NE approximability of exact KKT points, only working now with $\delta-$approximate (rather than exact) KKT points of (SMS).

Experimental Study of SYMMETRIC-2NASH Approximations

Experimental Evaluation of 2NASH Approximations

- Goal: Try various **heuristics** for providing approximate NE points in symmetric bimatrix games.
- Random Game Generator: Win-Lose symmetric games $\langle R, R^T \rangle$, provided by rounding a normalized-random game $\langle S, S^T \rangle$ whose entries are normal r.v.s with mean 0 and deviation 1. A **fine-tuning parameter** allows for favoring either sparse or dense win-lose games.
- OPTION: Clear the random win-lose game by avoiding...
  - "all-ones" rows in R, (weakly dominates all actions of row player).
  - "all-zeros" rows in R, (weakly dominated row which cannot disturb an approximate NE point).
  - $(1, 1)$–elements in the bimatrix $\langle R, R^T \rangle$ (trivial pure NE points).

1st Basic Approach

KKT Points of (SMS) as $\varepsilon-$NE Points

- Method: Use any polynomial-time construction algorithm to converge to a KKT point of SMS.
- Our approach: KKTSMS Use the **quadprog** (active-set) method of MatLab to locate a KKT point of (SMS).

2nd Basic Approach

Reformulation-Linearization Relaxation of (SMS)

- Method: Create a (1st-level) LP relaxation of SMS, based on the RLT method of [Sherali-Adams (1998)] .
- Our approach: RLTSMS Solve the following relaxation:

$$
\begin{array}{rrcll}
\textbf{minimize} & \alpha - \sum_{i\in[n]} \sum_{j\in[n]} R_{i,j} W_{i,j} & & & \\
\text{s.t.} & \beta - \sum_j (R_{i,j} + R_{k,j})\gamma_j + \sum_j \sum_\ell R_{i,j} R_{k,\ell} W_{j,\ell} & \geq & 0, & i,k \in [n] \\
& \alpha - \sum_j R_{i,j} x_j - \gamma_k + \sum_j R_{i,j} W_{j,k} & \geq & 0, & i,k \in [n] \\
& -x_i - x_j + W_{i,j} & \geq & -1, & i,j \in [n] \\
& \gamma_k - \sum_j R_{i,j}\gamma_j & \geq & 0, & i,k \in [n] \\
& x_i - \sum_j W_{i,j} & = & 0, & i \in [n] \\
& \sum_j x_j & = & 1, & \\
& -x_i - \alpha + \gamma_i & \geq & -1, & i \in [n] \\
& x_i - \gamma_i & \geq & 0, & i \in [n] \\
& \beta - \sum_j R_{i,j}\gamma_j & \geq & 0, & i \in [n] \\
& \sum_i \gamma_i - \alpha & = & 0, & \\
& \alpha - \beta & \geq & 0, & \\
\end{array}
$$

$$\beta \geq 0; \quad \gamma_i \geq 0,\ i \in [n]; \quad W_{i,j} \geq 0,\ i,j \in [n]$$

3rd Basic Approach

Doubly Positive SDP Relaxation of (SMS)

- Method: Create an SDP-relaxation of (SMS), considering also the non-negativity of the produced matrix.
- Our approach: DPSDP Solve the following relaxation:

$$
\begin{array}{lrcll}
\textbf{minimize} & \multicolumn{3}{c}{\alpha - \sum_i \sum_j R_{i,j} W_{i,j}} & \\
\text{s.t} & \alpha - \sum_j R_{i,j} x_j & \geq & 0, & i \in [n] \\
 & \sum_j x_j & = & 1, & \\
 & W_{i,j} - W_{j,i} & = & 0, & i,j \in [n] \\
 & Z \equiv \begin{bmatrix} W & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} & \succcurlyeq & 0, & \\
 & Z & \leq & \mathbf{1} \cdot \mathbf{1}^T, & \\
 & Z & \geq & \mathbf{0} \cdot \mathbf{0}^T &
\end{array}
$$

4th Basic Approach

Marginals of Extreme CE points of (SMS)

- Method: Return the marginals of extreme points in the CE-polytope of $\langle R, R^T \rangle$.
- Our approach:BMXCEV4 Solve the following relaxation and return the profile with the marginals of the optimum correlated strategy $\mathbf{W} \in \Delta([n]) \times \Delta([n])$:

$$\begin{array}{lrrcl} \textbf{min.} & & \sum_i \sum_j [R_{i,j} \cdot R_{j,i}] W_{i,j} & & \\ \text{s.t.} & \forall i,k \in [m], & \sum_{j \in [n]} (R_{i,j} - R_{k,j}) W_{i,j} & \geq & 0 \\ & & \sum_{i \in [m]} \sum_{j \in [n]} W_{i,j} & = & 1 \\ & \forall (i,j) \in [m] \times [n], & W_{i,j} & \geq & 0 \end{array}$$

Hybrid Approaches

- Method: Consider only best-of results for various (couples, or triples of) methods.

Experimental Results for Pure Methods (I)

	RLTSMS	KKTSMS	DPSDP	BMXCEV4
Worst-case ε	0.512432	0.22222	0.6	0.49836
#unsolved games	112999	110070	0	405
Worst-case round	10950	15484	16690	12139

- Experimental results for worst-case approximation among **500K** random **10x10** symmetric win-lose games.

Experimental Results for Pure Methods (II)

	RLTSMS	KKTSMS	DPSDP	BMXCEV4
Worst-case ε	0.41835	0.08333	0.51313	0.21203
#unsolved games	11183	32195	0	1553
Worst-case round	45062	42043	55555	17923

- Experimental results for worst-case approximation among **500K** random **10x10** symmetric win-lose games, which *avoid* $(1, 1)-$elements, $(1, *)-$ and $(0, *)-$rows.

Experimental Results for Hybrid Methods (I)

	KKTSMS + BMXCEV4	KKTSMS + RLT + BMXCEV4	KKTSMS + RLT + DPSDP
Worst-case ε	0.45881	0.47881	0.54999
#unsolved games	0	0	0
Worst-case round	652	1776	737

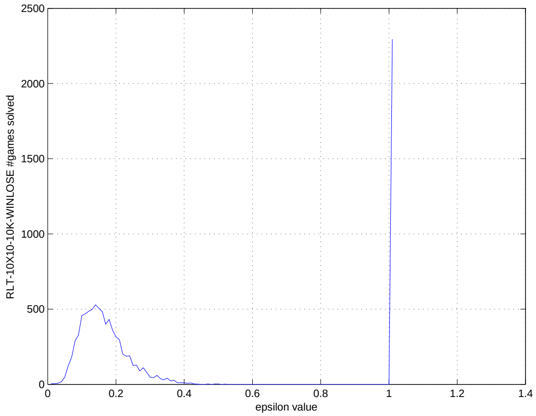
- Experimental results for worst-case approximation among **500K** random **10x10** symmetric win-lose games.

Experimental Results for Hybrid Methods (II)

	KKTSMS + BMXCEV4	KKTSMS + RLT + BMXCEV4	KKTSMS + RLT + DPSDP
Worst-case ε	0.08576	0.08576	0.28847
#unsolved games	0	0	0
Worst-case round	157185	397418	186519

- Experimental results for worst-case approximation among **500K** random **10x10** symmetric win-lose games, which *avoid* $(1,1)-$elements, $(1,*)-$ and $(0,*)-$rows.

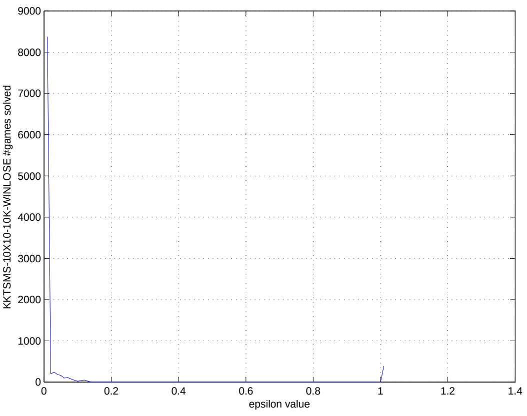
Distribution of Solved Games (I)



(a) RLTSMS

- Distribution of games solved for particular values of approximation, in runs of **10K** random **10x10** symmetric win-lose games. The games that remain unsolved in each case accumulate at the epsilon value 1.

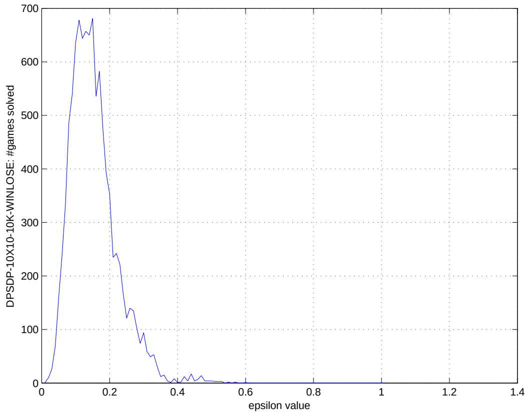
Distribution of Solved Games (II)



(b) KKTSMS

- Distribution of games solved for particular values of approximation, in runs of **10K** random **10x10** symmetric win-lose games. The games that remain unsolved in each case accumulate at the epsilon value 1.

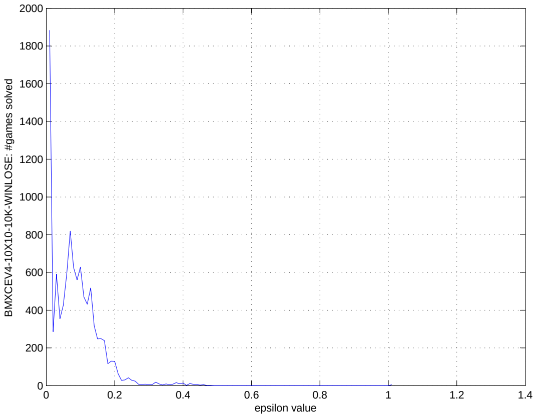
Distribution of Solved Games (III)



(c) DPSDP

- Distribution of games solved for particular values of approximation, in runs of **10K** random **10x10** symmetric win-lose games. The games that remain unsolved in each case accumulate at the epsilon value 1.

Distribution of Solved Games (IV)



(d) BMXCEV4

- Distribution of games solved for particular values of approximation, in runs of **10K** random **10x10** symmetric win-lose games. The games that remain unsolved in each case accumulate at the epsilon value 1.

Skeleton of the Talk

1. Bimatrx Games Preliminaries
   - Complexity of 2NASH
   - Formulations for 2NASH
   - The algorithm of Lemke & Howson
2. Polynomial-time Tractable Subclasses
3. Approximability of 2NASH
   - Theoretical Analysis
   - Experimental Study
4. Conclusions

Recap & Open Problems

- [Papadimitriou (2001)] kNASH is, together with FACTORING, probably the most important problems at the intersection of $\mathcal{P}$ and $\mathcal{NP}$.
- Even 2NASH seems already too hard to solve.
- Is there a PTAS, or a *lower bound* that, unless something extremely unlikely holds (eg, $\mathcal{P} = \mathcal{NP}$), excludes the existence of a better approximation ratio for ε-NE points?
  - Extensive experimentation on randomly constructed win-lose games shows that probably $1/3$ is not the end of the story...
- How about ε-WSNE points?
- Are there any other, more general subclasses of bimatrix game for which 2NASH is polynomial-time tractable, or at least a PTAS exists?

Thanks for your attention!

Questions / Remarks ?